3 DISTRIBUTION

3.1 BINOMIAL DISTRIBUTION

Some times we need to find the probability of x successes among n trials. Under the following assumptions we can find the required probability.

Assumptions :

1. There are only two possible outcomes for each trial (arbitrarily called 'success' and failure)
2. The probability of a success is the same for each trial.
3. There are n trials where n is a constant.
4. The n trials are independent.

Bernoulli trials Definition :

Trials satisfying these assumptions are called Bernoulli trials. Let E be an event of Bernoulli trial. Let probability of E i.e., probability of success is p. Let the probability of failure be q. Then $p + q = 1$, since Bernoulli trial is having only two possibilities of outcome success and failure sample space $= (E \cup E^c)$

Suppose there are n trials. Then probability of getting x successes and n – x failures is $p^x q^{n-x}$.

x factors of p, (n – x) factors of q are multiplied together by virtue of the generalized multiplication rule for more than two independent events.

This probability applies to any element of the sample space that represents x successes and n – x failures. Therefore we have to consider the number of combinations of x objects selected from a set of n objects.

The number of combinations of x objects selected among n objects is nC_x

∴ The probability of getting x successes and n – x failures among n trials is

$nC_x\, p^x q^{n-x}$

$$= nC_x\, p^x (1-p)^{n-x}$$

Binomial distribution :

This probability distribution is called the binomial distribution

$$B(x, n, p) = nC_x\, p^x (1-p)^{n-x}$$ (J NTU 2001)

2020.07.14 21:32

This distribution is called binomial distribution since the probabilities are the successive terms of the binomial expansion $(p+q)^n = p + (1-p)^n$

Example: **When n = 2**

Suppose two experiments can be thought of as a single experiment, the following four possibilities of the occurance of E and E^c exist.

1) EE (E occurs in the first and second trials)
2) EE^c (E occurs and E does not occurs)
3) E^cE (E does not occure and E occur)
4) E^cE^c (E does not occur and E does not occur)

Probabilities of four possibilities p^2, pq, qp, q^2.

Sum of the probabilities $p^2+2pq+q^2=(p+q)^2=1$, If X is a random variable

$P(X=0)=q^2, P(X=1)=2q, P(X=2)=p^2$

$\therefore (p+q)^n = nC_0 q^n + nC_1 pq^{n-1} + \ldots nC_n p^n$ and probability distribution is

x	0	1	...	...	...	n
p (x)	$nC_0 q^x$	$nC_1 pq^{n-1}$	...	...	...	p^n

WORKED OUT PROBLEMS

PROBLEM 1. ***A coin is tossed 9 times. Find the probability of getting 5 heads.***

Solution : $B(x, n, p) = nC_x p^x (1-p)^{n-x}$

here n = number of trials = 9.

$x = 5$, p = probability of getting head $= \frac{1}{2}$

$\therefore$ Required probability $= 9C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{9-5} = 9C_5 \left(\frac{1}{2}\right)^9 = \frac{63}{256}$

PROBLEM 2. ***A die is thrown 8 times. If getting a 2 or 4 is a success. Find the probability of***

i) ***4 success***

ii) $P(X \le 3)$

iii) $P(X \ge 2)$

Solution : $B(x, n, p) = nC_x P^x (1-p)^{n-x}$

n = 8 = number of trials = 8

p = probability of getting 2 or 4 $= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

i) $x = 4$

$$p(X=4) = 8C_4\left(\frac{1}{3}\right)^4\left(1-\frac{1}{3}\right)^{8-4} = 8C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^4 = \frac{1120}{6561}$$

ii) $p(X \le 3) = p(X=0) + p(X=1) + p(X=2) + p(X=3)$

$$= 8C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^8 + 8C_1\left(\frac{1}{3}\right)\cdot\left(\frac{2}{3}\right)^7$$

$$+ 8C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^6 + 8C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^5 = \frac{4864}{6561}$$

iii) $p(X \ge 2) = 1 - p(X=0) - p(X=1)$

$$= 1 - \left(\frac{2}{3}\right)^8 - 8\cdot\frac{2^7}{3^8} = \frac{5281}{6561}$$

PROBLEM 3. ***Two dice are thrown 5 times. If getting a doublet is a success. Find the probability that getting the success atleast once.***

Solution : $B(x, n, p) = nC_x\, p^x q^{n-x}$

Here $n = 5$, p = Probability of getting success.

S = sample space consists of 36 elements among which double line

(1, 1) (2, 2) (3, 3) (4, 4) (5, 5) (6, 6)

Total - 6.

$$\therefore \quad p = \frac{6}{36} = \frac{1}{6} \qquad \therefore \quad q = \frac{5}{6}$$

$P(X \ge 1) = 1 - P(X=0)$

$$= 1 - \left(\frac{5}{6}\right)^5 = \frac{4651}{7776}$$

PROBLEM 4. ***Among the items produced in a factory 5% are defective. Find the probability that a sample of 8 contains***

(i) exactly 2 defective items

(ii) greater than or equal to 7 defective items

(iii) at least one defective item.

Solution : $B(x, n, p) = nC_x\, p^x (1-p)^{n-x}$

Here $n = 8$

$$P = \text{probability of defective item} = \frac{5}{100} = \frac{1}{20}$$

i) $x = 2$

$$8C_2 \cdot \left(\frac{1}{20}\right)^2 \left(\frac{19}{20}\right)^6 = \frac{7}{100} \cdot \left(\frac{19}{20}\right)^6$$

ii) $P(X \geq 7) = P(X = 7) + P(X = 8)$

$$= 8C_7 \left(\frac{1}{20}\right)^7 \cdot \frac{19}{20} + \left(\frac{1}{20}\right)^8 = \frac{153}{(20)^8}$$

iii) p (atleast one defective item) = $P(X \geq 1)$

$$= 1 - P(X = 0) = 1 - \left(\frac{19}{20}\right)^8 = \frac{1 - (19)^8}{(20)^8}$$

PROBLEM 5. ***Six cards are drawn from a pack of 52 cards. Find the probability that***
(i) At least 3 are diamonds
(ii) only 4 are diamonds
(iii) none is a diamond.

Solution : $B(x, n, p) = nC_x\, p^x (1 - p)^{n-x}$
Here n = 6

P = probability of getting a diamond = $\frac{13}{52} = \frac{1}{4}$

i) $P(X \geq 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2)$

$$= 1 - 6C_0 \left(\frac{3}{4}\right)^6 - 6C_1 \left(\frac{1}{4}\right) \cdot \left(\frac{3}{4}\right)^5 - 6C_2 \left(\frac{1}{4}\right)^2$$

$$= 1 - \frac{3^6}{4^6} - 6 \cdot \frac{3^5}{4^6} - 15 \cdot \frac{3^4}{4^6} = \frac{694}{4096}$$

ii) $P(X = 4) = 6C_4 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^2 = \frac{135}{4096}$

iii) $P(X = 0) = \frac{3^6}{4^6} = \frac{729}{4096}$

PROBLEM 6. ***In a family of 5 children, find the probability that there are***
(i) 2 boys
(ii) At least one boy
(iii) All are boys
(iv) No boys

Solution : $B(x, n, p) = nC_x\, p^x (1 - p)^{n-x}$
Here n = number of children = 5

p = probability of getting a boy = $\frac{1}{2}$

(we assume that probability of getting a boy = $\frac{1}{2}$ and a girl = $\frac{1}{2}$)

i) $x = 2$

Required probability $= 5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 = \frac{10}{2^5} = \frac{5}{16}$

ii) $P(x \geq 1) = 1 - P(x = 0)$

$$= 1 - nC_0\, p^o (1 - p)^n = 1 - \left(\frac{1}{2}\right)^5 = 1 - \frac{1}{32} = \frac{31}{32}$$

iii) $P(x = 5) = 5C_5 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

iv) $P(x = 0) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

PROBLEM 7. ***In the above PROBLEM if there are 1000 families in a village, find the number of families that contain***

(i) ***2 boys each***
(ii) ***At least one boy each***
(iii) ***All are boys***
(iv) ***No of boys.***

Solution : Suppose N = The number of families who will be having x number of children, then
$N = p(X) \cdot 1000$ (Total no. of families)

$\therefore$ The number of families having 2 boys each.

$N = 1000 \cdot p(x = 2) = 1000 \cdot \frac{5}{16} = 312.5$

$= 313$ (since it is no of families it should be an integer)

ii) $N = p(x \geq 1) \cdot 1000 = \frac{31}{32} \cdot 1000 = 969$ families

iii) $N = p(x = 5) \cdot 1000 = \frac{1}{32} \cdot 1000 = 31$ families.

iv) $p(x = 0) \cdot 1000 = \frac{1}{32} \cdot 1000 = 31$ families

2020.07.14 21:35

PROBLEM 8. ***It has been claimed that in 60% of all solar heat installations the utility reduced by at least one third. Accordingly what are the probabilities the utility bill will be reduced by at least one - third in***

i) form of five installations ***(JNTU 2001***

ii) at least four of five installations

Solution : The probability that for solar heat installations the utility bill is reduced by

$p = \cdot 6$ (given)

$n = 5$

i) $B(x, n, p) = nC_x\, p^x (1-p)^{n-x}$

$x = 4$

Required probability $= 5C_4 (\cdot 6)^4 (\cdot 4)^{5-4} = \cdot 259$

ii) $x = 5,\ n = 5$

$B(5, 5, \cdot 6) = 5C_5\ (\cdot 6)^5 = \cdot 078$

Required probability $= p(x \geq 4) = p(x = 4) + p(x = 5) = \cdot 259 + \cdot 078$

PROBLEM 9. ***The probability that a man aged 60 will live to be 70 is ·6. What is probability that out of 9 men 60 at least 6 will live to be 70 ?***

Solution : The probability that a man aged 60 will live to be 70 is $p = \cdot 6 = \frac{3}{5}$

$n = 9$

Required probability $= P(x \geq 6) = P(x = 6) + P(x = 7) + P(x = 8) + P($

$B(x, n, p) = nC_x\, p^x (1-p)^{n-x}$

$$p(x \geq 6) = 9C_6\left(\frac{3}{5}\right)^6\left(\frac{2}{5}\right)^3 + 9C_7\left(\frac{3}{5}\right)^7\left(\frac{2}{5}\right)^2$$

$$+ 9C_8\left(\frac{3}{5}\right)^8 \cdot \frac{2}{5} + 9C_9\left(\frac{3}{5}\right)^9 \cdot \left(\frac{2}{5}\right)^0 = \frac{(129}{5^9}$$

PROBLEM 10. ***The probability that a man hitting a target is*** $\frac{1}{3}$

a) If he fires 6 times, what is the probability of hitting

(i) at the most 5 times

(ii) at least 5 times

(iii) exactly once

b) If he fires so that the probability of his hitting the target at least o greater than $\frac{3}{4}$***, find n.*** ***(JNTU 2007***

Solution : a) Here probability of hitting the target is $\frac{1}{3} = P$

$n = 6$

2020.07.14 21:35

i) $P(x \leq 5) = 1 - P(x = 6) = 1 - 6C_6 \cdot \left(\frac{1}{3}\right)^6 = 1 - \frac{1}{729} = \frac{728}{729}$

ii) $P(x \geq 5) = P(x = 5) + P(x = 6)$

$$= 6C_5 \left(\frac{1}{3}\right)^5 \frac{2}{3} + 6C_6 \left(\frac{1}{3}\right)^6 = \frac{13}{729}$$

iii) $P(x = 1) = 6C_1 \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^5 = \frac{192}{729}$

b) $P(x \geq 1) > \frac{3}{4}$

$$1 - P(x = 0) > \frac{3}{4}$$

$$1 - \left(\frac{2}{3}\right)^n > \frac{3}{4}, \frac{1}{4} > \left(\frac{2}{3}\right)^n$$

By trial we get n = 4. ∴ He must fire 4 times so that the probability of hitting the target at least once is greater than $\frac{3}{4} = n$.

PROBLEM 11. ***A box contains 9 cards numbered 1 to 9. If four cards are drawn with replacement. What is the probability that none is 1.***

Solution : Probability of getting one is $p = \frac{1}{9}$

$B(x, n, p) = n\,C_x\, p^x q^{n-x}$

$q = \frac{8}{9}$; $n = 4$

$$P(x = 0) = 4\,C_0 \left(\frac{1}{9}\right)^0 \cdot \left(\frac{8}{9}\right)^4 = \left(\frac{8}{9}\right)^4$$

PROBLEM 12. ***An insurance agent accepts policies of 5 men all of identical age and in good health. The probability that a man of this age will be alive 30 years is $\frac{2}{3}$. Find the probability that in 30 years***

(i) all five men.

(ii) at least one man

(iii) at most three will be alive.

Solution : $n = 5, p = \frac{2}{3}$

i) $B(x, n, p) = n\,C_x p^x (1 - p)^{n-x}$

$$P(x = 5) = 5\,C_5 \left(\frac{2}{3}\right)^5 = \frac{32}{243}$$

ii) P (x = at least one) = 1 – probability of no. one being alive

$$P(x=0) = 5C_0\left(\frac{1}{3}\right)^5 = \frac{1}{243}$$

$$\therefore \quad P(x \geq 1) = 1 - P(x=0) = 1 - \frac{1}{243} = \frac{242}{243}$$

iii) $P(x \leq 3)\ 1 - P(x > 3) \quad = 1 - [P(x=4) + P(x=5)]$

$$= 1 - \left[5C_4\left(\frac{2}{3}\right)^4 \frac{1}{3} + 5C_5\left(\frac{2}{5}\right)^5\right]$$

$$= 1 - \left[\frac{80}{243} + \frac{32}{243}\right] = \frac{131}{243}$$

PROBLEM 13. ***The Probability that the life of a bulb is 100 days is .05. Find the pr[obability] that out of 6 bulbs***

i) at least one

ii) none

iii) greater than 4

iv) between 1 and 3 will be leaving a life of 100 days. ($1 \leq x \leq 3$)

Solution : Probability that a bulb is of 100 days life is $\frac{1}{20} = p$

$n = 6$

$B(x, n, p) = n\,C_x P^x (1-p)^{n-x}$

i) $P(x \geq 1) = 1 - P(x=0) = 1 - 6\,C_o\left(\frac{1}{20}\right)^o \cdot \left(\frac{19}{20}\right)^6 = 1 - \left(\frac{1}{2}\right.$

ii) $P(x = 0) = \left(\frac{19}{20}\right)^6$

iii) $P(x > 4) \ = P(x=5) + P(x=6)$

$$= 6C_5\left(\frac{1}{20}\right)^5 \cdot \frac{19}{20} + 6C_6 \cdot \left(\frac{1}{20}\right)^6$$

$$= \left(\frac{1}{20}\right)^6 [114 + 1] = \frac{115}{(20)^6}$$

iv) $P(1 \leq x \leq 3) = P(x=1) + P(x=2) + P(x=3)$

$$= 6C_1\left(\frac{1}{20}\right) \cdot \left(\frac{19}{20}\right)^5 + 6C_2\left(\frac{1}{20}\right)^2\left(\frac{19}{20}\right)^4 + 6C_3\left(\frac{1}{20}\right)^3$$

$$= \frac{(19)^3}{(20)^6}[2471]$$

2020.07.14 21:35

PROBLEM 14. ***Find the maximum n such that the probability of getting no head in tossing a coin n times is greater than .1.*** ***(JNTU 2005 S, 2006 Nov, 2007 Feb)***

Solution : P = probability of getting head is = $\frac{1}{2}$

given that $P(x=0) > \frac{1}{10}$

$$P(x=0) = nC_0\, p^0 (1-p)^n > \frac{1}{10}$$

$$\left(\frac{1}{2}\right)^n > \frac{1}{10}$$

For n = 1 the probability (x = 0) = ·5

n = 2 ,, .25

n = 3 ,, .125

$n \geq 4$ probability is < .125

$\therefore$ n = 3

PROBLEM 15. ***Two dice are thrown 4 times. Getting a sum of 7 on the faces is a success. Find the probability that sum 7 gets***

(i) twice

(ii) never

(iii) only once.

Solution : P = probability of getting a sum 7 when two dice are thrown.

Sample space consists of 36 elements.

The event of getting a sum of 7 is [(1, 6) (6, 1) (2, 5) (5, 2) (3, 4) (4, 3)]

$\therefore$ probability of getting a sum of 7 is $\frac{6}{36} = \frac{1}{6}$

$\therefore$ $p = \frac{1}{6}$, n = 4.

i) $p(x=2) = 4C_2\left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^2 = \frac{150}{1296} = \frac{25}{216}$

ii) $p(x=0) = 4C_0\left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$

iii) $p(x=1) = 4C_1\left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^3 = \frac{500}{1296}$

PROBLEM 16. ***Assuming that half the population are consumers of rice. If 8 individu... taken to test find the probability that***

(i) two are consumers of rice

(ii) at least 2 are consumers of rice

(iii) $1 \le x \le 4$ are consumers of rice.

Solution : Here half of the population are assumed to be consumers of rice.

$\therefore$ probability that they are consumers of rice $= P = \frac{1}{2}$

$n = 8$

i) $P(x = 2) = 8C_2 \left(\frac{1}{2}\right)^8 = \frac{28}{256} = \frac{7}{64}$

ii) $P(x \ge 2) = 1 - P(x = 0) - P(x = 1)$

$$= 1 - 8C_0 \left(\frac{1}{2}\right)^8 - 8.\frac{1}{2^8} = \frac{247}{256}$$

iii) $P(1 \le x \le 4) = P(x = 1) + P(x = 2) + P(x = 3) + P(x = 4)$

$$= (8C_1 + 8C_2 + 8C_3 + 8C_4)\frac{1}{2^8} = \frac{162}{256}$$

PROBLEM 17. ***The ratio to the probabilities of 3 successes and 2 successes amo... independent trials = $\frac{1}{3}$, find the probability p of getting success.***

Solution : $B(x, n, p) = n\,C_x\, p^x (1 - p)^{n - x}$

$n = 5$ ratio when $x = 3$ and $x = 2$

$$= \frac{5C_3\, p^3 (1 - p)^2}{5C_2\, p^2 (1 - p)^3} = \frac{1}{3} \text{ given}$$

$5C_3 = 5C_2$

$\therefore \frac{p}{1 - p} = \frac{1}{3}$; $3p = 1 - p$; $4p = 1$ or $p = \frac{1}{4}$

$\therefore$ The probability of getting success of a trial is .25

PROBLEM 18. ***In a Binomial distribution if n = number of trials = 8, given that 4p (x... p (x = 3), find the value of p. (probability of getting a success)***

Solution : $n = 8$; $p(x = 5) = 8C_5\, p^5 (1 - p)^3$

$p(x = 3) = 8C_3\, p^3 (1 - p)^5$

given that $4p(x = 5) = p(x = 3)$

$4.8C_5\, p^5 (1-p)^3 = 8C_3\, p^3 (1-p)^5$

$4.\, p^2 = (1-p)^2$ since $8C_5 = 8C_3$

$4p^2 = 1 + p^2 - 2p,\ 3p^2 + 2p - 1 = 0$

$p = \frac{-2 \pm 4}{6} = \frac{1}{3}$ since negative value cannot be taken. P is not negative

$\therefore$ probability of getting success $p = \frac{1}{3}$

PROBLEM 19. ***A student takes a true false examination consisting of 8 questions. He guesses each answer. The guesses are made at random. Find the smallest value of n that the probability of guessing at least n correct answers is less than $\frac{1}{2}$.*** **(2005)**

Solution : $P(x \geq n) < \frac{1}{2}$

Probability of guessing a question $= P = \frac{1}{2}$

$n = 8$

$P(x \geq n) = 1 - P(x=0) - P(x=1) \ldots\ldots - P(x = n-1) < \frac{1}{2}$

ie $P(x=0) + P(x=1) + \ldots\ldots P(x=n-1) > \frac{1}{2}$

$$8C_0\left(\frac{1}{2}\right)^8 + 8C_1\left(\frac{1}{2}\right)^8 + 8C_2\left(\frac{1}{2}\right)^8 \ldots\ldots\ldots + 8C_{n-1}\left(\frac{1}{2}\right)^8 > \frac{1}{2}$$

$n - 1 = 3,\ \frac{1+8+28+56}{256} = \frac{93}{256} < \frac{1}{2}$

$n - 1 = 4,\ \frac{1+8+28+56+70}{256} = \frac{163}{256} > \frac{1}{2}$

$n - 1 = 4$ by trial

$\therefore\ n = 5$

$\therefore$ The least n such that $p(x \geq n) < \frac{1}{2}$ is 5

2020.07.14 21:36

PROBLEM 20. ***20% of items produced from a factory are defective. Find the probability in a sample of 5 chosen at random***

(i) none is defective

(ii) one is defective

(ii) $P(1 < x < 4)$ ***(JNTU 2005 S, 2006 Nov, 2007...)***

Solution : $n = 5$, p = probability that item is defective is $\frac{20}{100} = \frac{1}{5}$.

i) $p(x = 0) = 5C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^5 = \frac{1024}{3125}$

ii) $p(x = 1) = 5C_1 \frac{1}{5}\left(\frac{4}{5}\right)^4 = \frac{1280}{3125}$

$$p(1 < x < 4) = P(x = 2) + P(x = 3)$$

$$= 5C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^3 + 5C_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^2$$

$$= \frac{10}{3125}(64 + 16) = \frac{32}{125}$$

PROBLEM 21. ***Ten coins are thrown simultaneously. Find the probability of getting at least seven heads.*** ***(JNTU 1999, 2007 ...)***

Solution : p = probability of getting a head $= \frac{1}{2}$

$\therefore \quad q = \frac{1}{2}$

n = number of trials = 10

$$p(x \geq 7) = p(x = 7) + p(x = 8) + p(x = 9) + p(x = 10)$$

Binomial Distribution : The probability of getting x heads among n trials is

$$nC_x \, p^x q^{n-x}$$

$$p(x \geq 7) = 10C_7 \left(\frac{1}{2}\right)^{10} + 10C_8 \left(\frac{1}{2}\right)^{10} + 10C_9 \left(\frac{1}{2}\right)^{10} + 10C_{10} \left(\frac{1}{2}\right)^{10}$$

$$= \left(\frac{1}{2}\right)^{10} (10C_3 + 10C_2 + 10C_1 + 1)$$

$$= .1719$$

2020.07.14 21:36

PROBLEM 22. ***The probability of a man hitting a target is $\frac{1}{3}$***

i) If he fires 5 times what is the probability of his hitting the target at least twice.

ii) How many times must he fires so that the probability of his hitting the target at least once is more than 90%. (JNTU 2006)

Solution : The probability of hitting is $P = \frac{1}{3}$; n = 5

$$P(x \geq 2) = P(x=2) + P(x=3) + P(x=4) + P(x=5)$$

$$B(x, n, p) = n\,C_x\, p^x q^{n-x}$$

$$q = \frac{2}{3}$$

$$x = 2,\ P(x=2) = 5C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = \frac{80}{243}$$

$$x = 3,\ P(x=3) = 5C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = \frac{40}{243}$$

$$p(x=4) = {}^5C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right) = \frac{10}{243}$$

$$P(x=5) = \left(\frac{1}{3}\right)^5 = \frac{1}{243}$$

PROBLEM 23. ***Out of 800 families with 5 children each how many would you expect to have***
i) 3 boys (ii) either 2 or 3 boys (JNTU Sept. 2005)

Solution : Assume that the probability of getting a male child = Female child = $\frac{1}{2}$

i) $B(x, n, p) = nC_x\, p^x q^{n-x}$,

$$P(x=3) = 5C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{10}{32} = \frac{5}{16}$$

There are 800 families.

∴ The number of families having 3 boys.

$$= \frac{5}{16} \times 800 = 250$$

ii) P (x = 2 or 3)

$$P(x=2) = 5C_2\left(\frac{1}{2}\right)^5 = \frac{10}{32} = \frac{5}{16}$$

$$\text{The no. of families} = \frac{5}{10} \times 800 = 250$$

PROBLEM 24. ***Assume that 50% of all engineering students are good in Mathem... Determine the probabilities that among 18 engineering stude...***

(JNTU 20...)

i) exactly 10

ii) at least 10

iii) at most 8

iv) at least 2 and at most 9 are good in mathematics.

Solution : 50% of Engineering students are good

$$P = \frac{1}{2}, n = 18$$

i) $x = 10$

$$18C_{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{8} = \cdot 166$$

ii) $P(x \geq 10) = 1 - P(x=0) - P(x=1) - P(x=2) \ldots\ldots P(x=9)$

$$= 1 - \left(\frac{1}{2}\right)^{18} - 18\left(\frac{1}{2}\right)^{18} \ldots . 18C_9\left(\frac{1}{2}\right)^{18}$$

$$= \cdot 40$$

iii) $P(x \leq 8) = P(x=0) + P(x=1) + \ldots\ldots P(x=8)$

$$= \frac{1}{2^{18}}\left(18C_0 + 18C_1 + \ldots + 18C_8\right)$$

$$= \cdot 4072$$

iv) $P(2 \leq x \leq 9) = \frac{1}{2^{18}}\left(18C_2 + 18C_3 + \ldots + 18C_9\right)$

$$= \cdot 5926$$

Mean of Binomial Distribution

(JNTU 1999, 2000, 2002, 2007 Feb)

Among n trials, the probability of getting x successes is $nC_x\, p^x q^{n-x}$

$$\text{Mean} = \sum_{x=0}^{n} x \cdot nC_x\, p^x q^{n-x}$$

$$= nC_1\, p \cdot q^{n-1} + 2nC_2\, p^2 q^{n-2} + 3 \cdot nC_3\, p^3 q^{n-3} + \dots\dots\dots n \cdot nC_n\, p^n$$

$$= nC_1\, p \cdot q^{n-1} + \frac{2n(n-1)}{1 \cdot 2} p^2 q^{n-2} + 3 \cdot \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} p^3 q^{n-3} + \dots\dots \frac{n \cdot n(n-1)\dots 1}{1 \cdot 2 \cdot 3 \dots n} p^n$$

$$= np\,[\,q^{n-1} + (n-1)\,p\,q^{n-2} + \frac{(n-1)(n-2)}{1 \cdot 2} p^2 q^{n-3} + \dots \frac{(n-1)\dots\dots 1}{\lfloor n-1} \cdot p^{n-1}\,]$$

$$= np\,(p+q)^{n-1} = np \text{ since } p+q=1$$

$\therefore$ The mean of Binomial Distribution = n p

Variance of Binomial Distribution

(JNTU 1999, 2000, 2002, 2007 Feb, April)

$$\text{Variance} = \Sigma x^2 p(x) - \mu^2$$

$$= \sum_{x=0}^{n} nC_x\, p^x q^{n-x} x^2 - \mu^2$$

$$= \sum_{x=0}^{n} nC_x\, p^x q^{n-x}\,[\,x(x-1) + x\,] - \mu^2$$

$$= 1 \cdot 2\, nC_2\, p^2 q^{n-2} + 3 \cdot 2\, p^3 q^{n-3} + \dots nC_n\, n(n-1)\, p^n + \Sigma\, nC_x \cdot x\, p^x q^{n-x} - \mu^2$$

$$= n(n-1)\,p^2 \sum_{x=2}^{n} n-2C_{x-2}\, p^{x-2} q^{n-x} + np - n^2p^2$$

$$[\text{ Since } \Sigma x \cdot nC_x\, p^x q^{n-1} = \mu = np]$$

$$= n(n-1)\,p^2 (p+q)^{n-2} + np - n^2p^2 = n(n-1)\,p^2 + np - n^2p^2$$

$$(\because p+q=1)$$

$$= n^2p^2 - np^2 + np - n^2p^2 = np(1-p) = npq$$

$\therefore$ Variance of Binomial Distribution is npq.

Mode of Binomial Distribution

Mode is the value of x at which p (x) has maximum value.

The probability of getting x successes among n trials is $nC_x\, p^x q^{n-x}$.

Let x be the mode of Binomial distribution.

$\therefore \quad p(x) \geq p(x+1),\ p(x) \geq p(x-1)$

$p(x) \geq p(x+1)$

$$\therefore \quad \frac{p(x)}{p(x+1)} \geq 1$$

$$\text{i.e.} \quad \frac{nC_x\, p^x q^{n-x}}{nC_{x+1}\, p^{x+1} q^{n-x-1}}$$

$$p(x+1) = nC_{x+1}\, p^{x+1} q^{n-x-1} = \frac{n(n-1)(\ldots\ldots n-x)}{\lfloor x+1}\cdot p^{x+1}\cdot q^{n-x-1}$$

$$\frac{p(x)}{p(x+1)} = \frac{n(n-1)\ldots\ldots(n-x+1)}{\lfloor x}\, p^x q^{n-x}\,.\lfloor x+1$$

$$n(n-1)\ldots\ldots(n-x)\, p^{x+1} q^{n\text{-}\ldots}$$

$$= \frac{(x+1)}{(n-x)}\frac{q}{p} \geq 1$$

$(x+1)q \geq (n-x)p$

$qx + q \geq np - px$

$px + qx \geq np - q$

$(p+q)x \geq np - (1-p)$

$x \geq np - 1 + p$

$x \geq (n+1)p - 1 \quad \ldots\ldots\ldots\ldots (1)$

Taking $p(x) \geq p(x-1)$ we will get $x \leq (n+1)p \quad \ldots\ldots\ldots\ldots (2)$

from (1) and (2) $(n+1)p - 1 \leq x \leq (n+1)p$

Case (i) : If $(n+1)p$ is not an integer, Mode is the integral part of $(n+1)p$

In this case the distribution is called unimodal.

Case (ii) : If $(n+1)p$ is an integer both $(n+1)p$ and $(n+1)p-1$ will represent modes and the distribution is bimodal.

NOTE : *The recurrance formula for Binomial distribution is*

$$p(x+1) = \frac{(n-x)}{(x+1)}\frac{p}{q}\, p(x)$$

WORKED OUT PROBLEMS

PROBLEM 1. ***Determine the Binomial distribution for which the mean is 4 and variance 3 and find its mode.*** **(JNTU 1999)**

Solution : Given that mean = 4 = np

Variance is 3 = npq

$\therefore \quad q = \frac{3}{4} \qquad \therefore \quad p = \frac{1}{4} \qquad \therefore \quad n = \frac{4}{p} = 16$

Binomial distribution is $\left(\frac{1}{4} + \frac{3}{4}\right)^{16}$

Mode = $(n+1)p = \frac{17}{4}$ p

$(n+1)p$ is not an integer $\therefore$ Integral part of $(n+1)p$ i.e., 4 is the mode.

PROBLEM 2. ***For a binomial distribution mean = 6 and variance is 2. Find the probability of two successes.***

Solution : Mean = n p = 6, Variance npq = 2

$\therefore \quad q = \frac{2}{6}, p = 1 - \frac{1}{3} = \frac{2}{3}$

$np = 6, p = \frac{2}{3} \quad \therefore n = \frac{6}{p} = 9$

Probability when x = 2

$$9C_2 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^7 = 36 \cdot \frac{4}{3^9} = \frac{16}{3^7}$$

PROBLEM 3. ***In a Binomial distribution mean = 4 and variance is 2. Find the mode of Binomial distribution.***

Solution : Mean = np = 4

Variance npq = 2

$\therefore \quad q = \frac{2}{4} = \frac{1}{2} \qquad \therefore \quad p = 1 - q = \frac{1}{2}$

np = 4

$p = \frac{1}{2}$

$\therefore \quad n = \frac{4}{p} = 8$

$\therefore$ Mode $= (n+1)p = 9 \cdot \frac{1}{2}$ not an integer

$\therefore$ Mode $= 4$

The distribution is uniModal.

PROBLEM 4. ***Six dice are thrown 243 times. How many times do you expect at least t... to show a 5 or 6.***

Solution : $n = 6, N = 243$

The probability p of getting 5 or 6 is $\frac{1}{3}$

$p(x \geq 2) = 1 - p(x = 0) - p(x = 1)$

$p(x)$ = probability of getting n successes among n successes is $nC_x\, p^x q^{...}$

$p(x \geq 2) = 1 - nC_0\, p^0 q^n - nC_1\, p^1 q^{n-1}$

$$= 1 - \left(\frac{2}{3}\right)^6 - 6 \cdot \frac{1}{3} \cdot \left(\frac{2}{3}\right)^5 = 1 - \frac{2^6}{3^6}(1 + 3) = \frac{473}{729}$$

Expected number of dice $= N\,p(x) = 243 \cdot \frac{473}{729} = \frac{473}{3} = 158$

PROBLEM 5. ***If the probability of a defective bolt is $\frac{1}{8}$. Find***

(i) the mean

(ii) the variance for the distribution of defective bolts of 640.

Solution : Mean = np $\qquad q = 1 - p = \frac{7}{8}, \quad n = 640$

$p = \frac{1}{8} \qquad \therefore \quad \mu = \text{Mean} = \frac{640}{8} = 80$

Variance $= 640 \times \frac{1}{8} \cdot \frac{7}{8} = 70$

PROBLEM 6. ***The Mean of Binomial distribution is 3 and variance is $\frac{9}{4}$. Find***

(i) the value of n

(ii) $p(x \geq 1)$

(iii) $p(x \leq 7)$

(iv) $p(1 \leq x < 6)$

Solution : i) Mean $= \mu = 3 = np$

Variance $\sigma^2 = npq = \frac{9}{4}$

2020.07.14 21:37

$\therefore\ q = \frac{9}{4 \cdot 3} = \frac{3}{4}$

$\therefore\ p = \frac{1}{4},\ n = 3 \cdot 4 = 12$

ii) $p(X \geq 1) = 1 - p(X = 0)$

$$= 1 - nC_x\, p^x q^{n-x} = 1 - \left(\frac{3}{4}\right)^{12}$$

iii) $p(x \leq 7) = p(x=0) + p(x=1) + p(x=2) + p(x=3) + p(x=4)$
$+ p(x=5) + p(x=6) + p(x=7)$

$$= \left(\frac{3}{4}\right)^{12} + 12 \cdot \frac{1}{4} \cdot \left(\frac{3}{4}\right)^{11} + 66 \cdot \left(\frac{1}{4}\right)^{2} \cdot \left(\frac{3}{4}\right)^{10}$$

$$+ 220 \cdot \left(\frac{1}{4}\right)^{3} \cdot \left(\frac{3}{4}\right)^{9} + 495 \cdot \left(\frac{1}{4}\right)^{4}\left(\frac{3}{4}\right)^{8}$$

$$+ 72\left(\frac{1}{4}\right)^{5}\left(\frac{3}{4}\right)^{7} + 84 \times \left(\frac{1}{4}\right)^{6}\left(\frac{3}{4}\right)^{6} + 72\left(\frac{1}{4}\right)^{7} \cdot \left(\frac{3}{4}\right)^{5}$$

$$\frac{3^5}{4^{12}}\left[3^7 + 12 \cdot 3^6 + 66 \cdot 3^5 + 220 \cdot 3^4 + 495 \cdot 3^3 + 72 \cdot 3^2 + 84 \cdot 3 + 72\right]$$

$$= \frac{3^7}{4^{12}}\left[243 + 972 + 1782 + 1980 + 1485 + 72 + 28 + 8\right]$$

$$= \frac{3^7}{4^{12}} \cdot (6570)$$

iv) $p(1 \leq x < 6) = p(x=1) + p(x=2) + p(x=3) + p(x=4) + p(x=5)$

$$= 12\,\frac{1}{4} \cdot \left(\frac{3}{4}\right)^{11} + 12C_2\left(\frac{1}{4}\right)^{2} \cdot \left(\frac{3}{4}\right)^{10} + 12C_3\left(\frac{1}{4}\right)^{3} \cdot \left(\frac{3}{4}\right)^{9}$$

$$+ 12C_4\left(\frac{1}{4}\right)^{4}\left(\frac{3}{4}\right)^{8} + 12C_5\left(\frac{1}{4}\right)^{5}\left(\frac{3}{4}\right)^{7}$$

$$= \frac{3^7}{4^{12}}\left[972 + 1782 + 1980 + 1485 + 72\right] = \frac{3^7}{4^{12}}(6291)$$

3.2 THE POISSON APPROXIMATION TO THE BINOMIAL DISTRIBUTION

When n is large and p is small, binomial probabilities are often approximated by means of the distribution which is called Poisson Distribution.

Poisson Distribution $= p(x, \mu) = \dfrac{\mu^x e^{-\mu}}{\lfloor x}$, $x = 0, 1, 2, \ldots\ldots$ where $\mu = np$

THEOREM : Limit $n \to \infty$ Binomial distribution approaches Poisson Distribution.

(JNTU 2001 S, 2004, 2005 S, 2007 Feb)

PROOF : Binomial Distribution $B(x, n, p) = n\,C_x\, p^x (1 - p)^{n-x}$

where p is the probability of getting x successes among n trials.

$B(x, n, p) = nC_x\, p^x (1 - p)^{n-x}$

$$= \frac{n(n-1)(n-2)\ldots(n-x+1)}{\lfloor x} p^x (1-p)^{n-x}$$

$$= \frac{np(np-p)(np-2p)\ldots\ldots(np-(x-1)p)(1-p)^{n-x}}{\lfloor x}$$

(taking each p into one factor)

$$= \frac{\mu\left(\mu - \frac{\mu}{n}\right)\left(\mu - \frac{2\mu}{n}\right)\ldots\left(\mu - \frac{(x-1)\mu}{n}\right)\left(1 - \frac{\mu}{n}\right)^n}{\lfloor x \left(1 - \frac{\mu}{n}\right)^x} \quad \text{(since } \mu = np\text{)}$$

Now taking the limit as $n \to \infty$

$$= \frac{\mu^x e^{-\mu}}{\lfloor x}\left(\underset{n\to\infty}{Lt} \frac{\mu}{n}, \frac{2\mu}{n} \ldots\ldots \frac{(x-1)\mu}{n} \to 0\right)$$

$$\underset{n\to\infty}{Lt}\left(1 - \frac{\mu}{n}\right)^n = e^{-\mu}$$

$$\underset{n\to\infty}{Lt}\left(1 - \frac{\mu}{n}\right)^x = 1$$

$= p(x, \mu)$ which is poisson distribution.

$\therefore \underset{n\to\infty}{Lt} B(x, n, p) = p(x, \mu)$

To prove that $\sum_{x=0}^{\infty} p(x,\mu) = 1$

PROOF : $\sum_{x=0}^{\infty} p(x,\mu) = \sum_{x=0}^{\infty} e^{-\mu} \cdot \frac{\mu^x}{\lfloor x}$

$$= e^{-\mu}\left[1 + \mu + \frac{\mu^2}{\lfloor 2} + \frac{\mu^3}{\lfloor 3} + \ldots\ldots\ldots \infty\right] = e^{-\mu} \cdot e^{\mu} = 1$$

$$\therefore \sum_{x=0}^{\infty} p(x,\mu) = 1$$

i.e., sum of the probabilities x = 0, 1 ……… ∞ is 1.

(JNTU 2006)

To find the mean of Poisson distribution

$$\text{Mean} = \sum_{x=0}^{\infty} x p(x,\mu) = \sum_{x=0}^{\infty} x \frac{e^{-\mu} \cdot \mu^x}{\lfloor x} = \sum_{x=1}^{\infty} x \frac{e^{-\mu} \cdot \mu^x}{\lfloor x-1}$$

$$= e^{-\mu}\left[\mu + \mu^2 + \frac{\mu^3}{\lfloor 2} + \frac{\mu^4}{\lfloor 3} + \ldots\ldots\ldots \infty\right]$$

$$= e^{-\mu} \cdot \mu\left[1 + \mu + \frac{\mu^2}{\lfloor 2} + \frac{\mu^3}{\lfloor 3} + \ldots\ldots\ldots \infty\right]$$

$$= \mu \cdot e^{-\mu} \cdot e^{\mu} = \mu$$

∴ The mean of Poisson distribution = μ

(JNTU 2006)

To find the variance of Poisson distribution

$$p(x,\mu) = e^{-\mu} \cdot \frac{\mu^x}{\lfloor x}$$

$$E(X) = \mu$$

$$E(X^2) = \sum_{x=0}^{\infty} x^2 p(x,\mu) = \sum_{x=0}^{\infty} x^2 \cdot e^{-\mu} \cdot \frac{\mu^x}{\lfloor x}$$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{[x(x-1) + x]\mu^x}{\lfloor x}$$

$$= e^{-\mu} \cdot \sum_{x=2}^{\infty} \frac{\mu^x}{\lfloor x-2} + \sum_{x=0}^{\infty} x \cdot \frac{e^{-\mu} \cdot \mu^x}{\lfloor x}$$

$$= \bar{e}^{\mu} \cdot \mu^2 \sum_{x=2}^{\infty} \frac{\mu^{x-2}}{\lfloor x-2} + E(X)$$

$$= \bar{e}^{\mu} \cdot \mu^2 \cdot e^{\mu} + \mu = \mu^2 + \mu.$$

$$Var(X) = \sigma^2 = E(X^2) - (E(x))^2 = \mu^2 + \mu - \mu^2 = \mu$$

∴ The variance of poisson distribution is μ.

∴ The Mean and Variance of Poisson distribution are equal.

Mode of Poisson distribution :

To find the mode of Poisson distribution.

We know that the mode is the value of x for which the probability $p(x)$ is maximum.

$$\therefore \; p(x) \geq p(x+1) \text{ and } p(x) \geq p(x-1)$$

Now $p(x) \geq p(x+1)$

$$p(x) \geq p(x+1) \Rightarrow \frac{\bar{e}^{\mu} \mu^x}{\lfloor x} \geq \frac{\bar{e}^{\mu} \mu^{x+1}}{\lfloor x+1}; \; 1 \geq \frac{\mu}{x+1}$$

$$\therefore \; \frac{x+1}{\mu} \geq 1$$

$$x + 1 \geq \mu$$

$$x \geq \mu - 1$$

Similarly if we consider $p(x) \geq p(x-1)$, we will get $x \leq \mu$

$$\therefore \; \mu - 1 \leq x \leq \mu.$$

∴ The mode of Poisson distribution lies between $\mu - 1$ and μ.

Case i : If μ is an integer and $\mu - 1$ is also an integer, μ and $\mu - 1$ both are modes.
In this case Poisson distribution is bimodal.

Case ii : If μ is not an integer, the mode of poisson distribution is integral part of μ. In this case the distribution is called unimodal.

WORKED OUT PROBLEMS

PROBLEM 1. ***If the probability that an individual suffers a bad reaction from a ... injection is ·003. Find the probability that out of 1000 individuals i) ... 3 ii) more than or equal to 2 individuals iii) None suffers from a bad ...***

Solution : $\mu = np$; $n = 1000$; $p = \cdot 003$; $\mu = 1000 \times \cdot 003 = 3$

i) $x = 3$

Probability that 3 suffers among 1000 is p (3, 3)

$$p(x, \mu) = \frac{\bar{e}^{\mu} \cdot \mu^{x}}{x!}$$

$$p(3, 3) = \frac{\bar{e}^{3} \cdot (3)^{3}}{3!} = \frac{9}{2}\bar{e}^{3} = \frac{9}{2}(\cdot 0498) = \cdot 2241$$

ii) $p(x \geq 2) = 1 - p(x = 0) - p(x = 1)$

$$= 1 - \bar{e}^{3} - \bar{e}^{3} \cdot 3 = 1 - e^{3}(4)$$

$$= 1 - 4 \times (\cdot 0498)$$

$$= 1 - \cdot 1992$$

$$= \cdot 8008$$

iii) $p(x = 0) = p(0, 3) = \bar{e}^{3} = \cdot 0498$

PROBLEM 2. ***2% of the items of a factory are defective. The items are packed in boxes ... is the probability that there will be*** ***(JNTU ...)***

i) 2 defective items

ii) atleast three defective items

iii) 2 < defective items < 5 in a box of 100 items.

Solution : $\mu = np$; $n = 100$, p = probability of defective items

$$= \frac{2}{100} = \cdot 02$$

$$\mu = \cdot 02 \times 100 = 2$$

i) $x = 2$

$$p(2, 2) = \bar{e}^{2} \cdot \frac{(2)^{2}}{2!} = 2\bar{e}^{2} = 2 \times (\cdot 136) = \cdot 272$$

ii) $p(x \geq 3) = 1 - p(x = 0) - p(x = 1) - p(x = 2)$

$$= 1 - \bar{e}^{2} - 2\bar{e}^{2} - \bar{e}^{2}\frac{(2)^{2}}{2} = 1 - 5\bar{e}^{2} = 1 - 5 \times \cdot 136 = \ldots$$

iii) $p(2 < x < 5) = p(x = 3) + p(x = 4)$

$$= \frac{\bar{e}^{2} \cdot (2)^{3}}{3!} + \frac{\bar{e}^{2} \cdot 2^{4}}{4!} = 2\bar{e}^{2} = \cdot 272$$

PROBLEM 3. ***The Probabilities of a Poisson variate taking the values 1 and 2 are equal. Calculate the probabilities of the variate taking the values 0 and 3.***

Solution : $p(x, \mu) = \dfrac{e^{-\mu}\ \mu^{x}}{x!}$

$$p(1, \mu) = p(2, \mu)$$

$$e^{-\mu} \cdot \mu = \frac{e^{-\mu} \cdot \mu^2}{2!} \quad \therefore \mu = 2$$

$$p(x = 0) = p(0, 2) = e^{-2} = \cdot 136$$

$$p(x = 3) = p(3, 2) = \frac{e^{-2} \cdot 2^3}{3!} = \frac{4e^{-2}}{3} = \cdot 181$$

PROBLEM 4. ***If for a poisson variate 2p (x = 0) = p (x = 2). Find the probability that***

i) $p(x \le 3)$,

ii) $p(2 < x \le 5)$

iii) $p(x \ge 3)$

Solution : $p(x, \mu) = \dfrac{e^{-\mu} \cdot \mu^{x}}{x!}$

Given that 2 × probability at x = 0 = probability at x = 2

$$\therefore \quad 2e^{-\mu} = e^{-\mu}\frac{\mu^2}{2!} \qquad \therefore \qquad \mu^2 = 4$$

$$\therefore \quad \mu = 2$$

i) $p(x \le 3) = p(x = 0) + p(x = 1) + p(x = 2) + p(x = 3)$

$$= e^{-2} + e^{-2} \cdot 2 + e^{-2} \cdot \frac{2^2}{2!} + e^{-2} \cdot \frac{2^3}{3!}$$

$$= e^{-2}\left(1 + 2 + 2 + \frac{4}{3}\right) = \frac{19}{3}e^{-2} = \cdot 0861$$

ii) $p(2 < x \le 5) = p(x = 3) + p(x = 4) + p(x = 5)$

$$= e^{-2} \cdot \frac{2^3}{3!} + e^{-2} \cdot \frac{2^4}{4!} + e^{-2} \cdot \frac{2^5}{5!} = \frac{34}{15}e^{-2} = \cdot 0308$$

iii) $p(x \ge 3) = 1 - p(x = 0) - p(x = 1) - p(x = 2)$

$$= 1 - e^{-2} - 2 \cdot e^{-2} - \frac{2^2}{2!}e^{-2} = 1 - 5e^{-2} = \cdot 32$$

2020.07.14 21:40

PROBLEM 5. ***If X is a poisson variate such that p (x = 1) = 24 p (x = 3). Find the probability p (x = 0)***

Solution : $p(x, \mu) = \dfrac{\bar{e}^{\mu} \cdot \mu^{x}}{x!}$

Given that p (x = 1) = 24 p (x = 3)

$$e^{-\mu} \cdot \mu = 24 \, \frac{\bar{e}^{\mu} \cdot \mu^{3}}{3!}, \quad \mu^{2} = \frac{6}{24} = \frac{1}{4}$$

$$\therefore \quad \mu = \frac{1}{2} = \cdot 5$$

$$p(x = 0) = \bar{e}^{\mu} = \bar{e}^{\cdot 5}$$

PROBLEM 6. ***One fifth percent of the blades produced by a blade manufacturing f... turnout to be defective. The blades are supplied in packets of 10. Use Po... distribution to calculate the approximate number of packets containi...***

i) no defective blade

ii) one defective blade in a consignment of 10000.

Solution : p = probability of defective blades $= \dfrac{1}{500}$

n = 10

$$\mu = np = 10 \cdot \frac{1}{500} = \frac{1}{50} = \cdot 02$$

Assuming that X is a Poisson variate

with $\mu = \cdot 02$

$$p(x) = \frac{\bar{e}^{\mu} \cdot \mu^{x}}{x!}$$

i) Probability of getting no defective blade is $p(x = 0) = \bar{e}^{\cdot 02} = \cdot 9802$

$\therefore$ The number of packets containing no defective blades is $10000 \times \cdot 9802 = 9802$

ii) Probability of getting one defective blade $p(x = 1) = \dfrac{\bar{e}^{\cdot 02} \cdot 02}{1!} = \cdot$...

$\therefore$ The number of packets which may contain one defective blade $= \cdot 0196 \times 10000 = 196$

2020.07.14 21:40

RECURRENCE FORMULA FOR THE POISSON DISTRIBUTION

We know that probability at any value of x when mean is given by μ is

$$p(x, \mu) = \frac{e^{-\mu} \cdot \mu^{x}}{x!} = \text{Poisson distribution}$$

$$p(x+1) = \frac{e^{-\mu} \cdot \mu^{x+1}}{x+1!} = \frac{e^{-\mu} \cdot \mu^{x} \cdot \mu}{x! \; x+1} = p(x) \frac{\mu}{(x+1)}$$

$$\therefore \quad p(x+1) = \frac{\mu}{(x+1)} p(x)$$

$\therefore$ The probability at x = x + 1 is $\frac{\mu}{(x+1)}$ times the probability at x.

◆ WORKED OUT PROBLEMS ◆

PROBLEM 16. ***If X is a poisson variate such that p (x = 0) = p (x = 1) find p (x = 0) and using recurrence formula find the probabilities at x = 1,2,3,4 and 5.(JNTU 2004 S)***

Solution : $p(x, \mu) = \frac{e^{-\mu} \cdot \mu^{x}}{x!}$

Given that probability at x = 0 = probability at x = 1

$$\therefore \quad e^{-\mu} = \mu \, e^{-\mu} \qquad \therefore \qquad \mu = 1$$

$$\therefore p(x=0) = e^{-\mu} = e^{-1} = \cdot 368 = p(x=1)$$

$$p(x+1) = \frac{\mu}{x+1} p(x)$$

put x = 1

$$p(x=2) = \frac{1}{2} p(1) = \frac{1}{2} \cdot 368 = \cdot 184$$

put x = 2

$$p(3) = \frac{1}{3} p(2) = \cdot 0613$$

$$p(4) = \frac{1}{4} p(3) = \cdot 0153$$

$$p(5) = \frac{1}{5} p(4) = \cdot 00306$$

$\therefore$ Probabilities at x = 0, 1, 2, 3, 4, 5 are ·368, · 368, ·184, ·0613, ·0153, ·00306

PROBLEM 17. ***Fit a Poisson distribution for the following data and calculate the expected frequencies*** (JNTU …)

x	0	1	2	3	4
f(x)	109	65	22	3	1

N = 200

Solution : Mean of Poisson distribution = μ.

To find μ

$$\mu = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{0 \times 109 + 1 \times 65 + 2 \times 22 + 3 \times 3 + 1 \times 4}{109 + 65 + 22 + 3 + 1} = \cdot 61$$

frequency $f(x) = N\,p(x) = 200\,p(x)$

$$p(0) = e^{-\mu} = e^{-\cdot 61} = \cdot 5435$$

$$f(0) = 200 \times \cdot 5435 = 109$$

$$f(1) = 200 \times p(1)$$

$$f(x+1) = N \cdot p(x+1) = \frac{N \cdot \mu}{x+1}\, p(x) = \frac{\mu}{x+1}\, f(x)$$

$$f(1) = \cdot 61\, f(0) = 109 \times \cdot 61 = 65$$

$$f(2) = \frac{\cdot 61}{2}\, f(1) = \cdot 305 \times 65 = 20$$

$$f(3) = \frac{\cdot 61}{3}\, f(2) = \frac{20 \times \cdot 61}{3} = 4$$

$$f(4) = \frac{\cdot 61}{4}\, f(3) = \frac{\cdot 61}{4} \times 4 = 1$$

expected frequencies are

x	0	1	2	3	4
f(x)	109	65	20	4	1

PROBLEM 18. ***The distribution of typing mistakes committed by a typist is given below assuming the distribution to be poisson, find the expected frequencies.***

x	0	1	2	3	4	5
f(x)	125	95	49	20	8	3

Solution : $\mu = \frac{\Sigma x f(x)}{\Sigma f(x)} = \frac{0 \times 125 + 1 \times 95 + 2 \times 49 + 3 \times 20 + 4 \times 8 + 5 \times 3}{125 + 95 + 49 + 20 + 8 + 3}$

$$= \frac{300}{300} = 1$$

$$p(x, \mu) = \frac{e^{-\mu} \cdot \mu^x}{x!}; N = \Sigma f(x) = 300$$

$$f(0) = N \times p(0) = N \cdot e^{-1} = 300 \times \cdot 368 = 110 \cdot 4, f(0) = f(1)$$

$$f(x+1) = \frac{\mu}{x+1} f(x),$$

$$\therefore \quad f(1) = 110 \cdot 4$$

$$f(2) = \frac{1}{2} \times 110 \cdot 4 = 55 \cdot 2$$

$$f(3) = \frac{1}{3} f(2) = \frac{1}{3} \times 55 \cdot 2 = 18 \cdot 4$$

$$f(4) = \frac{1}{4} f(3) = \frac{1}{4} \times 18 \cdot 4 = 4 \cdot 6$$

$$f(5) = \frac{1}{5} f(4) = 1$$

$\therefore$ expected frequencies are

x	0	1	2	3	4	5
f(x)	110·4	110·4	55·2	18·4	4·6	1

PROBLEM 19. ***Use Recurrence formula to find probabilities x = 0, 1, 2, 3, 4 and mean 3.***

Solution : $\mu = 3.\ p(x) = \dfrac{e^{-\mu} \cdot \mu^{x}}{x!}$

$$p(x+1) = \frac{\mu}{x+1}\, p(x)$$

when $x = 0$

$$p(0) = e^{-\mu} = e^{-3} = 0{\cdot}498$$

$$p(1) = \frac{3}{1}\, p(0) = 3 \times {\cdot}0498 = {\cdot}1494$$

$$p(2) = \frac{3}{2}\, p(1) = \frac{3}{2}({\cdot}1494) = {\cdot}2241$$

$$p(3) = \frac{3}{3}\, p(2) = p(2) = {\cdot}2241$$

$$p(4) = \frac{3}{4}\, p(3) = \frac{3}{4}({\cdot}2241) = {\cdot}1681$$

$$p(5) = \frac{3}{5}\, p(4) = {\cdot}1009$$

∴ The probabilities are

x	0	1	2	3	4	5
p (x)	·0498	·1494	·2241	·2241	·1681	·1009

PROBLEM 20. ***Fit a Poisson distribution and find the expected frequencies for the following data***

x	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
f (x)	*1*	*4*	*15*	*22*	*21*	*20*	*8*	*6*	*2*	*0*	*1*

Solution : $\Sigma f = N = 100$

∴ expected frequencies are

x	0	1	2	3	4	5	6	7	8	9	10
f (x)	1·8	7·2	14·4	19·2	19·2	15·36	10·24	5·87	2·94	1·3	·52

3.3 THE NORMAL DISTRIBUTION

Introduction :

The distributions Binomial and Poisson are discrete distributions whereas the normal distribution is the continuous distribution.

The distribution of the continuous random variable x is represented by f (x), where f (x) is the density function. The graph of density functions f (x) is continuous examples.

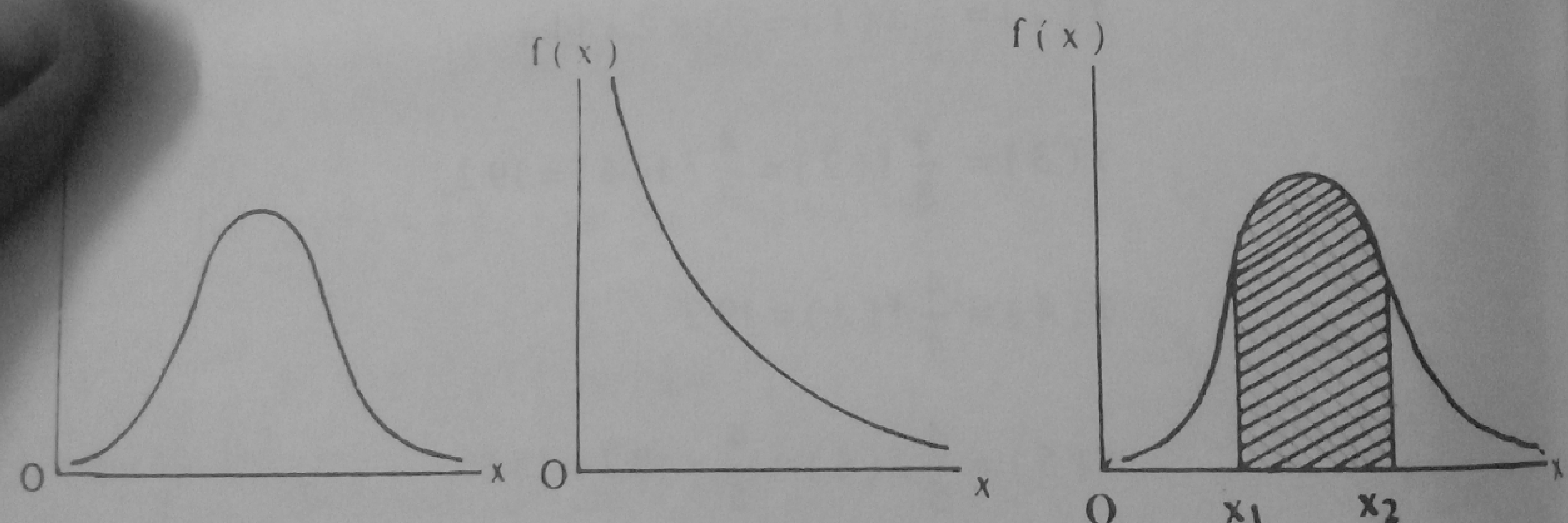


fig. 3.1

examples of graphs of probability density functions.

A Probability density function is constructed so that the area under its curve bounded by the x axis is 1, when computed over the range of x. The probability that x assumes the values between x_1 and x_2 is the shaded area in the figure.

$$\therefore p(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} f(x)\,dx \text{ and the total area } \int_{-\infty}^{\infty} f(x)\,dx = 1 \; (-\infty \leq x \leq \infty$$

Normal distribution :

The continuous random variable x is said to have a normal distribution, if its probability density function is defined by $f(x) = c\,e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}$ for $-\infty \le x \le \infty$ where c is a constant.

It involves two parameters a and b, the constant c is determined by the condition that total area under the curve is unity.

To find the value of c.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \qquad \therefore \int_{-\infty}^{\infty} c \cdot e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2} dx = 1$$

Put $\frac{x-a}{b} = z$

$$= cb\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = 1$$

$dx = bdz$

$$= 2\,cb\int_{0}^{\infty} e^{-\frac{z^2}{2}} dz = 1$$

since f (z) is even function.

$$= 2cb\int_{0}^{\infty} e^{-t}\frac{dt}{\sqrt{2}\sqrt{t}}$$

Put $\frac{z^2}{2} = t$

$$= \sqrt{2}\,cb\int_{0}^{\infty} e^{-t} \cdot t^{-\frac{1}{2}} dt$$

$2z\,\frac{dz}{2} = dt$

$$= \sqrt{2}\,c\,b\,\gamma\left(\frac{1}{2}\right) = \sqrt{2}\,c\,b\sqrt{\pi}$$

$dz = \frac{dt}{z}$

since $\gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$z = \sqrt{2t}$

$$\therefore \quad \sqrt{2\pi} \cdot c \cdot b = 1$$

$n - 1 = -\frac{1}{2}$

$n = \frac{1}{2}$

$$\therefore \quad c = \frac{1}{\sqrt{2\pi}\,b}$$

$$\therefore \quad f(x) = \frac{1}{\sqrt{2\pi}\,b}\,e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2} \text{ in } (-\infty \le x \le \infty)$$

To find the mean μ (JNTU 2004 S, 200...)

$$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\, b}\, e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\, b}\int_{-\infty}^{\infty} (bz + a)\, e^{-\frac{z^2}{2}}\, dz$$

Put $\frac{x-a}{b} = z \quad dx = b\,dz \quad x = bz + a$

$$= \frac{1}{\sqrt{2\pi}\, b}\, b\left[\int_{-\infty}^{\infty} z\, e^{-\frac{z^2}{2}} dz + a\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 + a\cdot 2\int_{0}^{\infty} e^{-\frac{z^2}{2}} dz\right] \quad \left(\because z\, e^{-\frac{z^2}{2}} \text{ is odd}\right)$$

$$= \frac{2a}{\sqrt{2\pi}}\cdot\frac{\sqrt{\pi}}{\sqrt{2}} = a \quad \left(\because \int_{0}^{\infty} e^{-\frac{z^2}{2}} dz = \frac{\sqrt{\pi}}{\sqrt{2}} \text{ above integration}\right)$$

∴ $\mu = a$ mean of normal distribution.

To find the variance σ^2 of the normal distribution.

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \frac{1}{\sqrt{2\pi}\, b}\int_{-\infty}^{\infty} (x-\mu)^2\, e^{-\frac{1}{2}\left(\frac{x-\mu}{b}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\, b}\int_{-\infty}^{\infty} b^2 z^2\, e^{-\frac{z^2}{2}} dz\cdot b \qquad \text{Put } \frac{x-\mu}{b} = z$$

$$= \frac{b^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^2\, e^{-\frac{z^2}{2}} dz \qquad dx = b\,dz$$

$$= \frac{2b^2}{\sqrt{2\pi}} \int_0^\infty z^2 \, e^{-\frac{z^2}{2}} \, dz \qquad \left(\because z^2 e^{-\frac{z^2}{2}} \text{ is even} \right)$$

$$= \frac{2b^2}{\sqrt{2\pi}} \int_0^\infty \sqrt{2t} \, e^{-t} \, dt \qquad \frac{z^2}{2} = t$$

$$= \frac{2b^2}{\sqrt{\pi}} \int_0^\infty e^{-t} \cdot t^{\frac{1}{2}} \, dt \qquad zdz = dt;\ z = \sqrt{2t}$$

$$n - 1 = \frac{1}{2};\ n = \frac{3}{2}$$

$$= \frac{2b^2}{\sqrt{\pi}} \gamma\left(\frac{3}{2}\right) = \frac{2b^2}{\sqrt{\pi}} \frac{1}{2} \gamma\left(\frac{1}{2}\right) \qquad \gamma(n+1) = n\gamma(n)$$

$$= \frac{b^2}{\sqrt{\pi}} \sqrt{\pi} \qquad \because \left(\frac{1}{2}\right) = \sqrt{\pi}$$

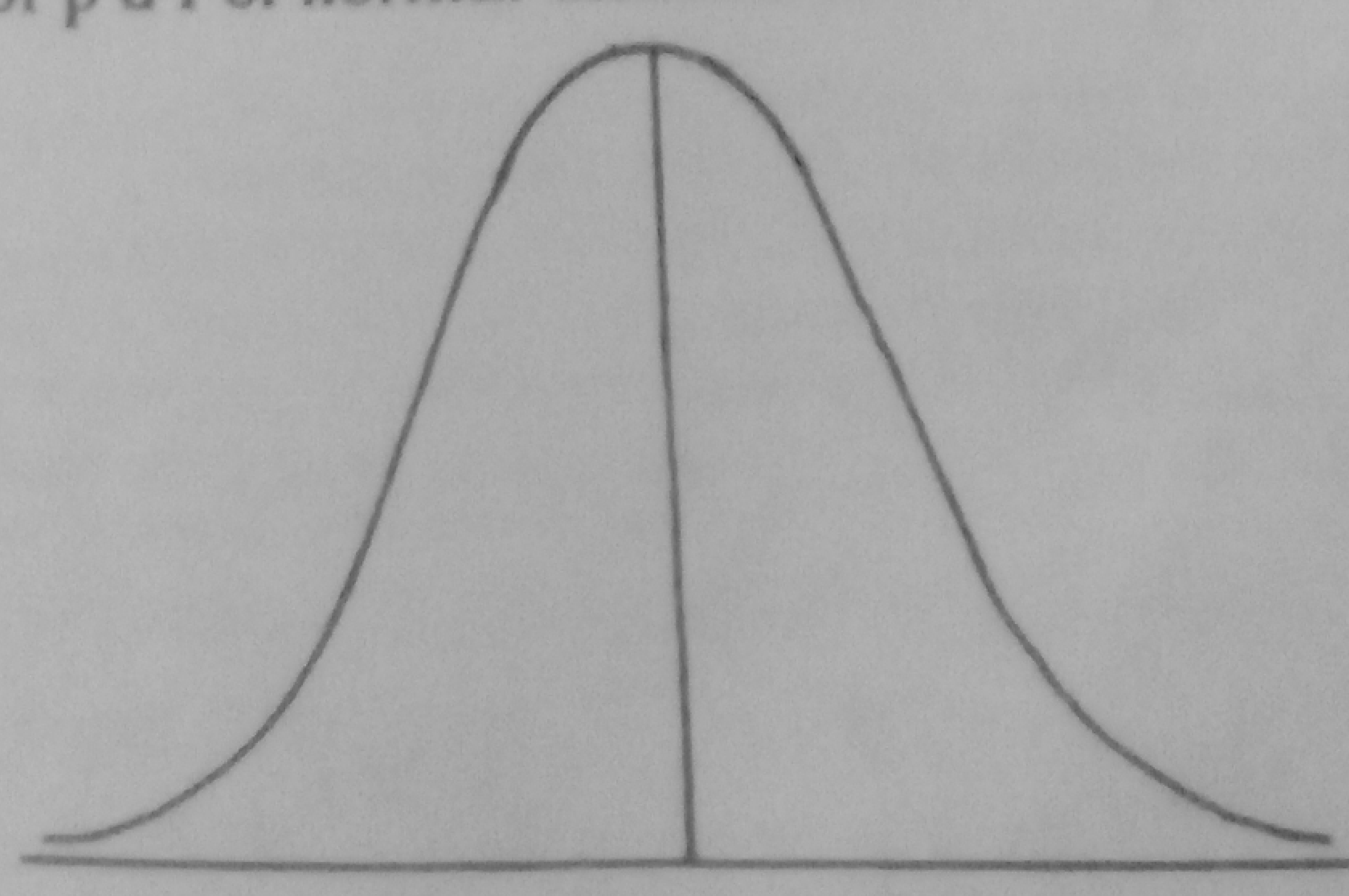
$\therefore \sigma^2 = b^2$

$\therefore$ Standard deviation $\sigma = b$

$\therefore$ The probability density function of the normal

distribution is $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2}$

The graph of p·d·f of normal distribution is



$x = \mu$

Fig 3.2

Mode of Normal Distribution : (JNTU 2004 S, 2007

Mode is the value of x for which f (x) is maximum.

$\therefore f'(x) = 0$ and $f''(x) = -$ ve at that value of x.

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$f'(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\left(-\left(\frac{x-\mu}{\sigma^2}\right)\right)$$

$$= 0 \text{ when } x = \mu \text{ since } e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \neq 0$$

$$f''(x) = \frac{1}{\sqrt{2\pi}\,\sigma^2}\left[-\frac{1}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} - \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma}\right] = -\text{ve whe}$$

$\therefore$ f (x) is maximum at $x = \mu$

$\therefore \mu$ is the mode of normal distribution.

Median of normal distribution (JNTU 2004 S, 2007

Suppose M is the median of normal distribution then $\int_{-\infty}^{M} f(x)\,dx = \int_{M}^{\infty} f(x)$

$$\int_{-\infty}^{M} f(x)\,dx = \frac{1}{2}$$

$$\text{i.e., } \int_{-\infty}^{M} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

$$\int_{-\infty}^{\mu} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \int_{\mu}^{M} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

Consider $\int_{-\infty}^{\mu} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$ Put $\frac{x-\mu}{\sigma} = z$

$$= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{z^2}{2}} \sigma\, dz$$ when $x = \mu,\ z = 0$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{z^2}{2}} dz \text{ (by symmetry)}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\sqrt{2}} = \frac{1}{2}$$

$$\therefore \frac{1}{2} + \int_{\mu}^{M} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

$$\therefore \int_{\mu}^{M} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 0$$

$$= (\text{if} \int_{a}^{b} f(x)\,dx = 0, \text{ then } a = b$$

$\therefore \mu = M$

$\therefore$ Median of the normal distribution is μ.

$\therefore$ For normal distribution mean = median = mode = μ

Points of inflexion of normal curve

The points of inflexion the values of x for which
$f''(x) = 0$ and $f'''(x) \neq 0$

$f''(x)$ can be written as $\frac{-f(x)}{\sigma^2}\left[1 - \left(\frac{x-\mu}{\sigma}\right)^2\right]$

2020.07.14 21:42

(see mode of normal distribution)

$$f''(x) = 0 \text{ when } 1 = \frac{x-\mu}{\sigma},\ x-\mu = \sigma$$

$$x = \mu + \sigma$$

$$f'''(x) = \frac{f(x)}{\sigma^4}\left[2(x-\mu)^2 + (x-\mu) - \frac{(x-\mu)^3}{\sigma^2}\right]$$

$$\neq 0 \text{ when } x-\mu = \sigma$$

$\therefore$ The points of inflexion of the normal curve are given by $x - \mu = \sigma$

Area Property (Normal Probability Integral)

If probability density function as normal distribution

$$\frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = N(\mu, \sigma^2)$$

The probability that random value of X will lie between $X = \mu$ and $X = x_1$ is given by

$$p(\mu < X < x_1) = \int_{\mu}^{x_1} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{x_1} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

Put $\frac{x-\mu}{\sigma} = Z$, $x - \mu = \sigma Z$, $dx = \sigma\, dZ$ limits are 0 to Z_1 where $Z_1 = \frac{x_1 - \mu}{\sigma}$

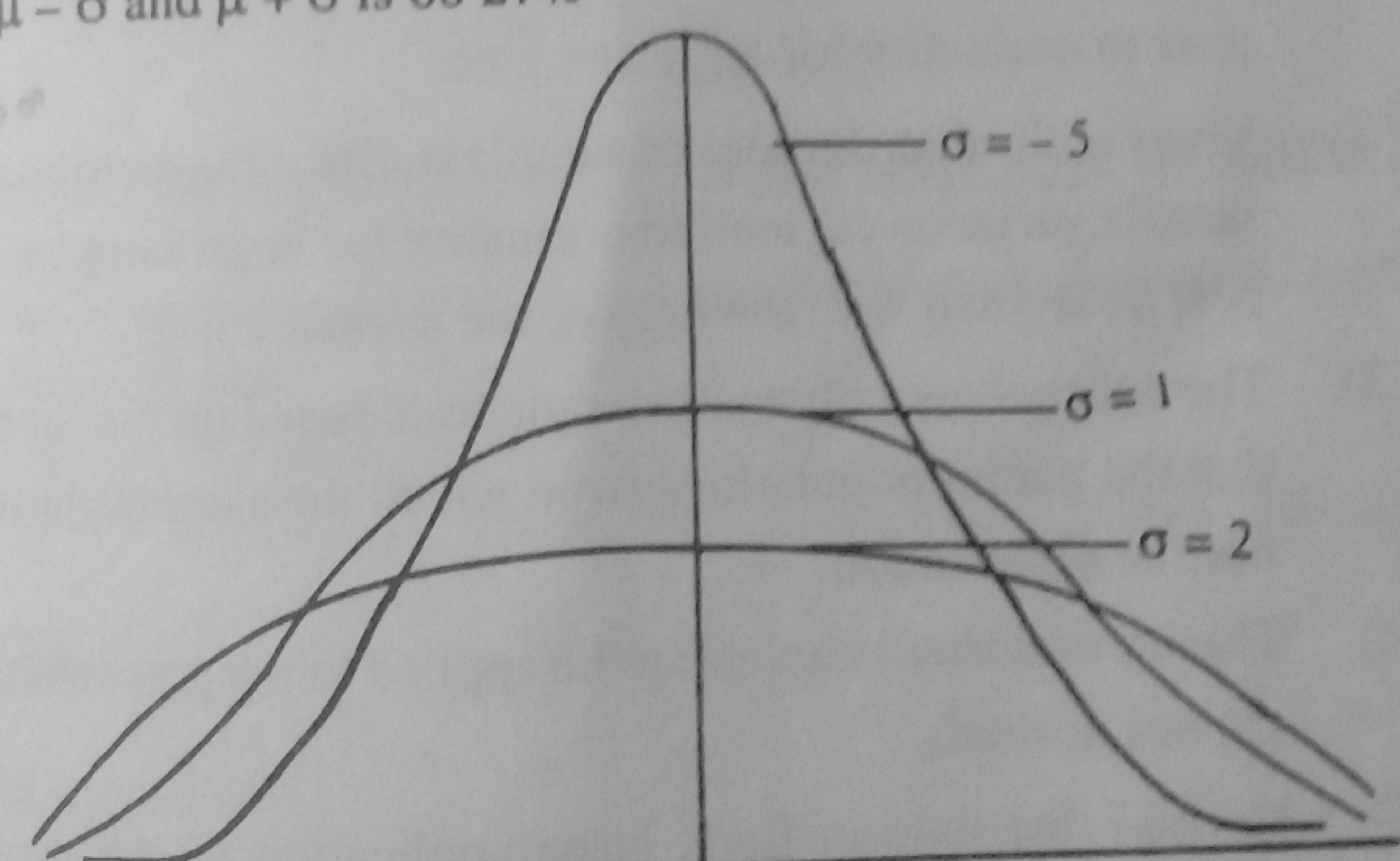
$$\frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-\frac{z^2}{2}}\,dZ = \int_0^{z_1} \phi(z)\,dz$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}$ is the probability function of standard normal variate. The definite integral $\int_a^b \phi(z)\,dz$ is known as normal probability integral and gives the area under standard normal curve between the ordinates $z = 0$ and $z = z_1$. These areas have been tabulated for different values of z_1 at intervals of 0.1

2020.07.14 21:42

Chief characteristics of normal distribution (JNTU 2004)

1. The mean, median and mode for normal distribution are identical.
2. The curve is smooth, regular, bell-shaped and symmetrical about the line $x = \mu$ since the curve $y = f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ does not have the odd powers of $\left(\frac{x-\mu}{\sigma}\right)$.
3. The ordinate of the curve decreases rapidly as $|x|$ increases, the maximum ordinate at $x = \mu$ is given by $y_{max} = \frac{1}{\sqrt{2\pi}\,\sigma}$ and the curve is unimodal.
4. The curve extends from $-\infty$ to ∞
5. As σ becomes larger, the ordinate y decreases i.e., the curve spreads out more but flattered at the top. On the other hand when σ becomes smaller y increases and the curve becomes more peaked.
6. Total area under the curve above x axis from $-\infty$ to ∞ is unity.
7. Area of normal curve between $\mu - \sigma$ and $\mu + \sigma$ is 68·27%



Normal Curve

fig. 3.3

8. Area between $\mu - 2\sigma$ and $\mu + 2\sigma$ is 95·45%
9. Area between $\mu - 3\sigma$ and $\mu + 3\sigma$ is 99·73%
10. The area bounded by the curve with x – axis and any two ordinates equals to the probability for the interval marked as x axis by the two ordinates.

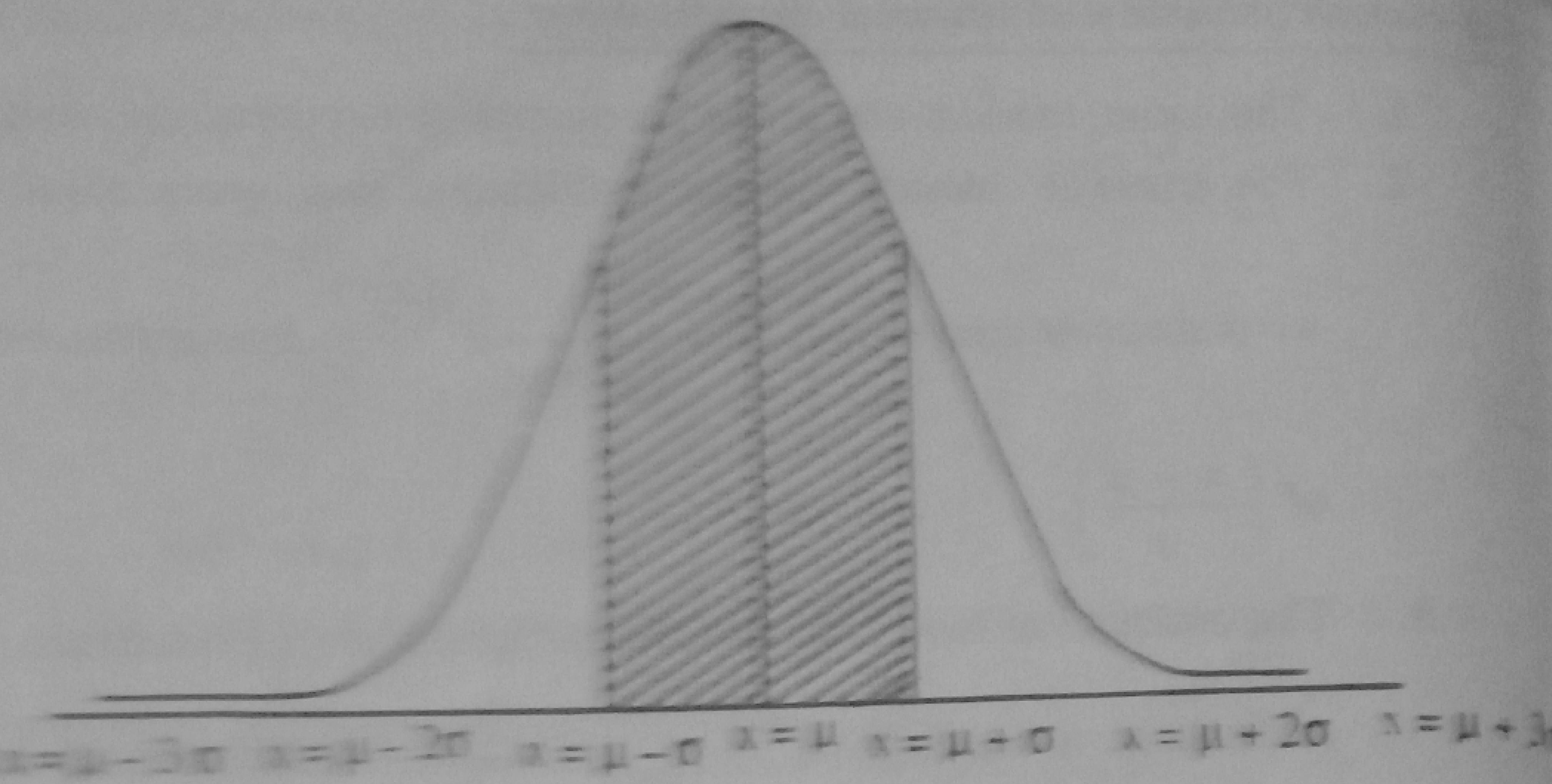


fig. 3.4

Importance of Normal Distribution (JNTU 1999, 2000, 2000 S, 2007

Normal distribution plays a very important role in statistical theory

(1) Most of the distributions occuring in practice example Binomial, Poi Hypergeometric distributions etc can be approximated by normal distri more over, many of the sampling distributions students χ, χ^2 distributio tend to normality for large samples.

(2) Many of the distributions of sample statistic, the distributions of sample m sample variance etc tend to normality for large samples and as such the best be studied with the help of the normal curve.

(3) The entire theory of small sample tests based on the fundamental assum that the parent populations from which the samples have been drawn is normal distribution.

(4) Theory of normal curves can be applied to the graduation of the curves w are not normal.

(5) Normal distribution finds large application in statistical quality cont industry for setting control limits.

Mean Deviation from the Mean for Normal Distribution.

$$\text{M.D. (about mean)} = \int_{-\infty}^{\infty} |x - \mu| \, f(x) \, dx$$ (JNTU 2001 S)

2020.07.14 21:42

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} |x-\mu|\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \quad \text{Put } \frac{x-\mu}{\sigma} = z$$

$$\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |z|\, e^{-\frac{z^2}{2}}\, dz \qquad dx = \sigma dz$$

$$= \frac{2\sigma}{\sqrt{2\pi}}\int_{0}^{\infty} |z|\, e^{-\frac{z^2}{2}}\, dz \quad \text{since } |z|\, e^{-\frac{z^2}{2}} \text{ is even}$$

$$= \sqrt{\frac{2}{\pi}}\,\sigma\int_{0}^{\infty} e^{-t}\, dt = \sqrt{\frac{2}{\pi}}\,\sigma\left(-e^{-t}\right)_0^{\infty} \qquad \frac{z^2}{2} = t, \quad zdz = dt$$

$$= \sqrt{\frac{2}{\pi}}\,\sigma = \frac{4}{5}\sigma \text{ (approximately)}$$

WORKED OUT PROBLEMS

PROBLEM 1. ***If X is a normal variate with mean 30 and standard deviation 5. Find the probabilities that***

i) 26 X 40 and

ii) X 45 ***(JNTU 2000, 2007 April)***

Solution : i) $z = \frac{x-\mu}{\sigma}$

μ = mean = 30

σ = standard deviation = 5

$z_1 = \frac{x_1-\mu}{\sigma}, \qquad x_1 = 26$

$z_1 = \frac{26-30}{5} = -\cdot 8$

$z_2 = \frac{x_2-\mu}{\sigma}, \qquad x_2 = 40$

$$z_2 = \frac{40 - 30}{5} = 2$$

$p(26 \le X \le 40) = p(-\cdot 8 \le z \le 2)$

$p(-\cdot 8 \le z \le 2)$ = Area EABCDE

= Area EOCDE + Area OABCO

$= p(-\cdot 8 \le z \le 0) + p(0 \le z \le 2)$

$= p(0 \le z \le \cdot 8) + p(0 \le z \le 2)$ by symmetry

$= \cdot 2881 + \cdot 4772 = \cdot 7653$ (from tables)

ii) $X \ge 45$

$$z = \frac{45 - 30}{5} = 3$$

$p(X \ge 45) = p(z \ge 3)$ shaded protion

in the curve = Area OCBD – Area OABCO

$= \cdot 5 -$ Area $(z = 3)$

$= \cdot 5 - \cdot 4987 = \cdot 0013$

$\therefore p(X \ge 45) = \cdot 0013$

PROBLEM 2. ***Find the probability that a random variable having the standard normal distribution will take on a value between ·87 and 1·28*** **(JNTU 2001)**

Solution : $z_1 = \cdot 87$, $z_2 = 1{\cdot}28$

$p(\cdot 87 \le z \le 1{\cdot}28)$

$= p(\text{at } z = 1{\cdot}28) - p(\text{at } 22{\cdot}87)$ (from tables)

$= \cdot 8997 - \cdot 8078 = \cdot 0919$

PROBLEM 3. ***Find the probability between x = 15 and 60. Given that Mean μ = 40 and standard deviation = 10.***

Solution : μ = Mean = 40

σ = Standard deviation = 10

$x_1 = 15;\ z = \dfrac{x - \mu}{\sigma}$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{15 - 40}{10} = -2{\cdot}5$$

$$z_2 = \frac{x_2 - \mu}{\sigma},\ x_2 = 60$$

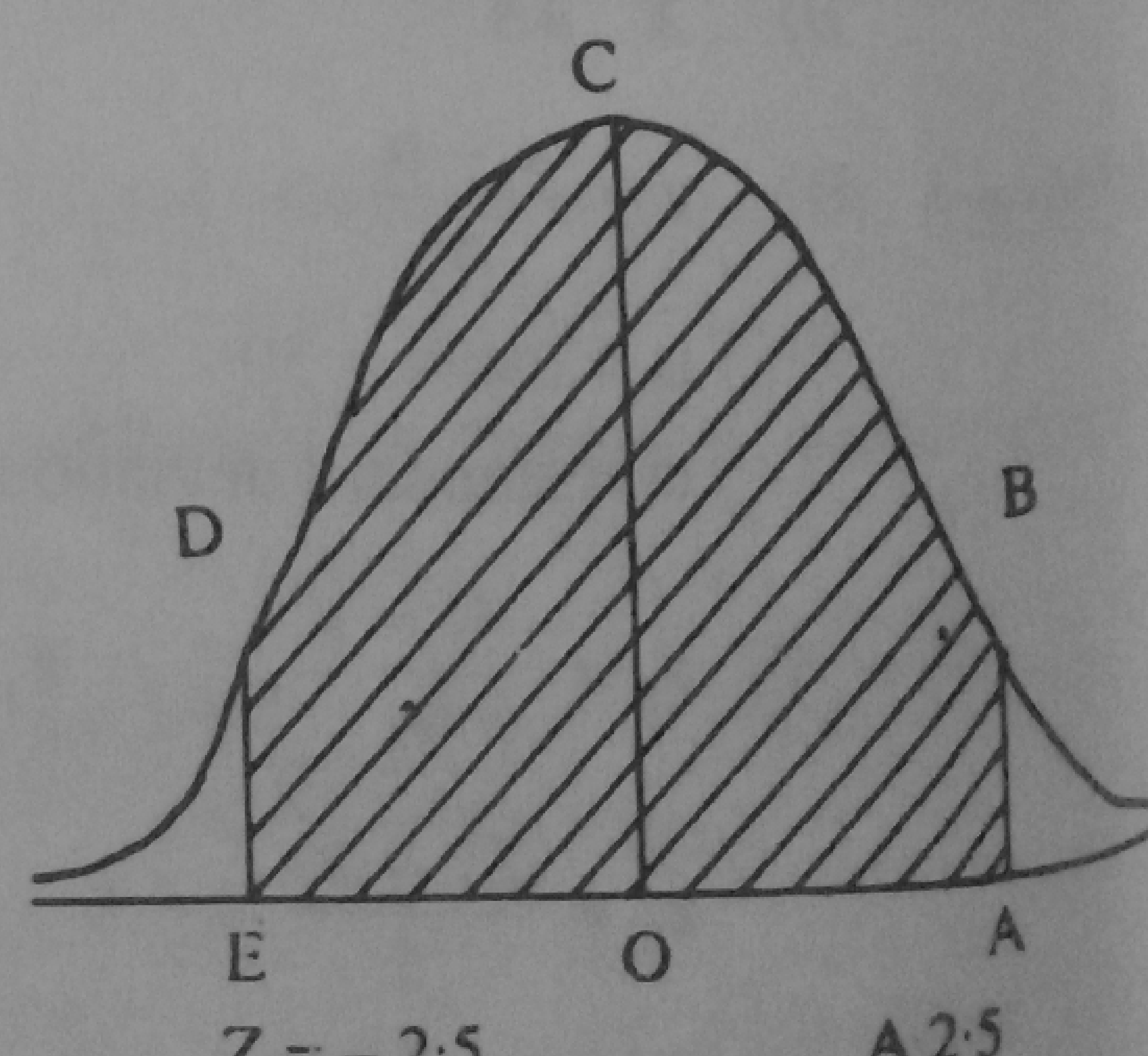


fig. 3.5

2020.07.14 21:43

$$z_2 = \frac{60 - 40}{10} = 2$$

$p(15 \le x \le 60) = p(-2{\cdot}5 \le z \le 2)$

$p(-{\cdot}25 \le z \le 2)$

= Area EOABCDE

= Area EOCDE + Area OABCO

$= p(-{\cdot}25 \le z \le 0) + p(0 \le z \le 2)$

$= p(0 \le z \le 2{\cdot}5) + p(0 \le z \le 2)$ by symmetry

$= {\cdot}4938 + {\cdot}4772 = {\cdot}9710$

PROBLEM 4. ***Given that the mean heights of students in a class is 158 cms with standard deviation of 20 cms. Find how many students heights lie between 150 cms and 170 cms if there are 100 students in the class.***

Solution : μ = Mean = 158 cms

σ = standard deviation = 20 cms

Probability of students whose heights lie between 150 cms and 170 cms

$= p(150 \le X \le 170)$

$x_1 = 150,\ x_2 = 170$

$$z = \frac{x - \mu}{\sigma}$$

$$z_1 = \frac{150 - 158}{20} = \frac{-2}{5} = -{\cdot}4$$

$$z_2 = \frac{170 - 158}{20} = \frac{3}{5} = {\cdot}6$$

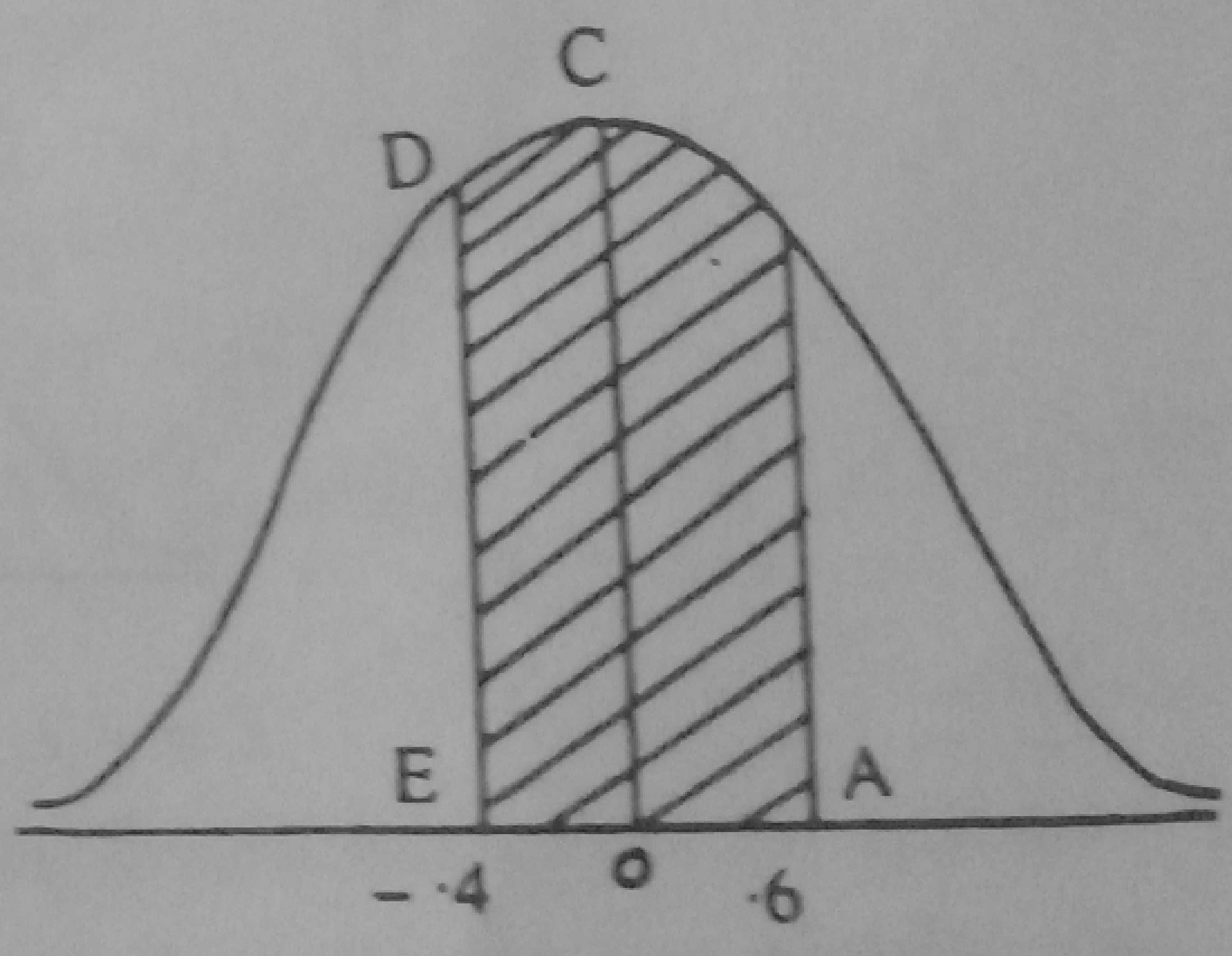


fig. 3.6

$p(150 \le x \le 170) = p(-{\cdot}4 \le z \le {\cdot}6)$

= Area EOABCDE

= Area EOCDE + OABCO

$= p(-{\cdot}4 \le z \le 0) + p(0 \le z \le {\cdot}6)$

$= p(0 \le z \le {\cdot}4) + p(0 \le z \le {\cdot}6)$ by symmetry

$= {\cdot}1554 + {\cdot}2257 = {\cdot}3811$

No. of students whose height lie between 150 cms and 170 cms

= probability × total no. of students

$= {\cdot}38 \times 100 = 38$ (Number of people should be integers)

2020.07.14 21:43

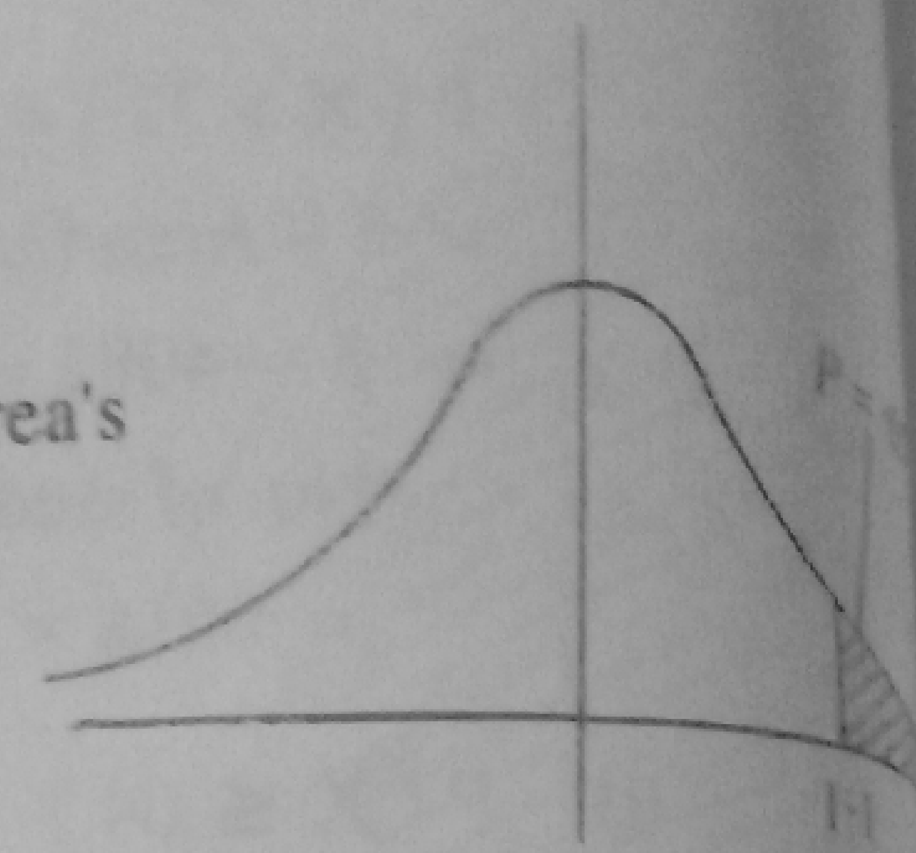
$$z = \frac{90 - 78}{11} = 1{\cdot}1$$

$P(x \geq 90) = P(z \geq 1{\cdot}1)$

${\cdot}5 - {\cdot}3643 = {\cdot}1357$,

No of student got 90% above

$= {\cdot}1354 \times 1000 = 136$



b) For lowest 10% the area is ·1 when the area's ·1 = · 5 – ·1 = · 4 Area OACD = · 4 when the area is post ·4 Z = 1·28

c) The area = ·9 when area ·45,Z = 1·64, Z = – 1·64 and Z = 1·64.

PROBLEM 19. ***Suppose the weights of 800 male students are normally distrib... with mean μ = 140 pounds and standard deviation 10 pounds. Find number of students whose weights are***

i) Between 138 and 148 pounds

ii) more than 152 pounds (JNTU 2006)

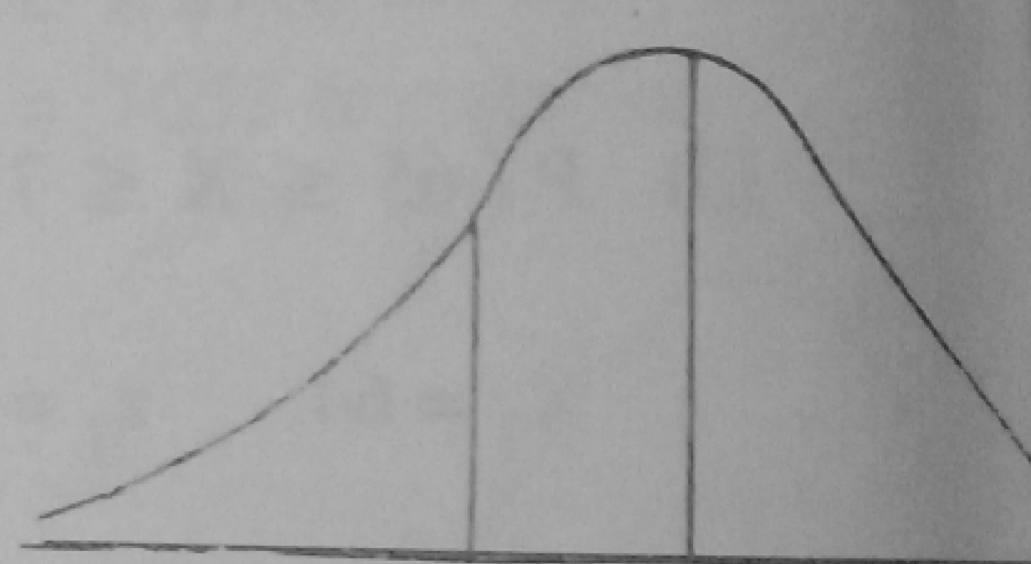
Solution : The mean of the normal distribution $\mu = 140$

The standard deviation $\sigma = 10$

$x = 138$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{138 - 140}{10} = -{\cdot}2$$

$$z_2 = \frac{148 - 140}{10} = {\cdot}8$$



$P(138 \leq x \leq 148)$

$= P(-2 \leq z \leq {\cdot}8)$

$= P(-{\cdot}2 \leq z \leq 0) + P(0 \leq z \leq {\cdot}8)$

$= {\cdot}0793 + {\cdot}2881 = {\cdot}3674$

The number of students whose weights lie between 138 and 148 is ·3674×800=

▶▶ 3.4 THE NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION

The normal distribution can be used to approximate the binomial distribution

Suppose the number of successes ranges from x_1 to x_2 i.e., we will have to the probability $x = x_1$ to $x = x_2$

i.e., $\sum_{x_1}^{x_2} nC_x p^x (1-p)^{n-x}$

When n is large for any class x the real class interval is $\left(x_1 - \frac{1}{2}, x_2 + \frac{1}{2}\right)$.

Therefore for x to rie in (x_1, x_2) we have to consider $\left(x_1 - \frac{1}{2}, x_2 + \frac{1}{2}\right)$

$p(x_1 \le x \le x_2) = p(z_1 \le z \le z_2)$

where $z_1 = \dfrac{x_1 - \frac{1}{2} - np}{\sqrt{npq}}$ and $z_2 = \dfrac{x_2 + \frac{1}{2} - np}{\sqrt{npq}}$

Therefore $p(x_1 \le x \le x_2) = \int_{z_2}^{z_1} \phi(z)\,dz$ which can be evaluated using normal tables.

WORKED OUT PROBLEMS

PROBLEM 1. ***If p is probability of getting head in tossing a coin and a coin is tossed 12 times find p (3 ≤ x ≤ 6) using***

i) binomial distribution

ii) Normal distribution.

Solution : i) $n = 12$ = number of trials

p = probability of getting a success $= \frac{1}{2}$

$p(3 \le x \le 6) = p(x=3) + p(x=4) + p(x=5) + p(x=6)$

$= 12C_3\left(\frac{1}{2}\right)^{12} + 12C_4\left(\frac{1}{2}\right)^{12} + 12C_5\left(\frac{1}{2}\right)^{12} + 12C_6\left(\frac{1}{2}\right)^{12}$

$= \left(\frac{1}{2}\right)^{12}(220 + 495 + 792 + 924) = \frac{2431}{4096} = \cdot 6$

ii) using normal distribution

Mean $= \mu = np = 12 \times \frac{1}{2} = 6$

$\sigma = \sqrt{npq} = \sqrt{12 \cdot \frac{1}{4}} = \sqrt{3} = 1\cdot 73$

$z_1 = \dfrac{3 - \frac{1}{2} - 6}{1\cdot 73} = \dfrac{2\cdot 5 - 6}{1\cdot 73} = -2\cdot 02$

$z_2 = \dfrac{6 + \frac{1}{2} - 6}{1\cdot 73} = \dfrac{6\cdot 5 - 6}{1\cdot 73} = \cdot 29$

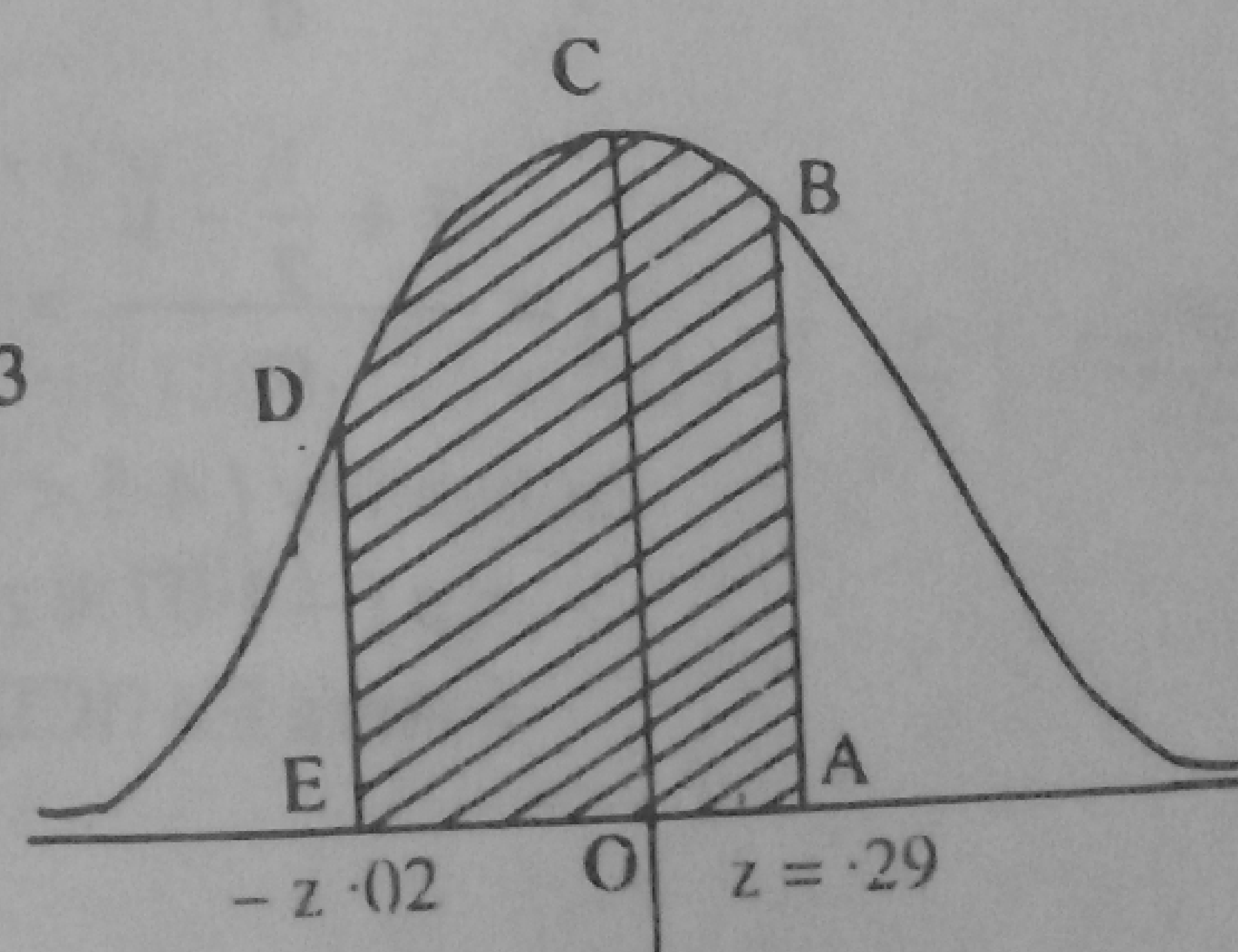


fig. 3.30

2020.07.14 21:43

$$p(3 \le x \le 6) = p(z_1 \le z \le z_2) = p(-2{\cdot}02 \le z \le {\cdot}29)$$

$$= \text{Area EABCDE} = \text{Area OABCO} + \text{OCDEO}$$

$$= p(-2{\cdot}02 \le z \le 0) + p(0 \le z \le {\cdot}29)$$

$$= p(0 \le z \le 2{\cdot}02) + p(0 \le z \le {\cdot}29)$$

$$= {\cdot}4782 + {\cdot}1141 = {\cdot}5923$$

PROBLEM 2. ***Find the probability of getting 1 or 4 or 5 or 6 in throwing a die 5 to 7 times among 9 trials using***

i) binomial distribution

ii) normal distribution.

Solution : i) p = probability of getting 1 or 4 or 5 or 6 in throwing a die $= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$ =

n = number of trials = 9

$$q = 1 - p = \frac{1}{3}$$

$$p(5 \le X \le 7) = p(X = 5) + p(X = 6) + p(X = 7)$$

$$= 9C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^4 + 9C_6 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^3 + 9C_7 \left(\frac{2}{3}\right)^7 \left(\frac{1}{3}\right)^2$$

$$= 126 \cdot \frac{2^5}{3^9} + 84 \cdot \frac{2^6}{3^9} + 36 \cdot \frac{2^7}{3^9} = \frac{2^5}{3^9}(126 + 168 + 144) = {\cdot}712$$

ii) **using normal distribution**

$$\mu = np = 9 \cdot \frac{2}{3} = 6$$

$$\sigma = \sqrt{npq} = \sqrt{9 \cdot \frac{2}{3} \cdot \frac{2}{3}} = \sqrt{2} = 1{\cdot}41$$

$$z_1 = \frac{5 - \frac{1}{2} - \mu}{\sigma} = \frac{4{\cdot}5 - 6}{1{\cdot}41} = -1{\cdot}07$$

$$z_2 = \frac{7 + \frac{1}{2} - \mu}{\sigma} = \frac{7{\cdot}5 - 6}{1{\cdot}41} = 1{\cdot}07$$

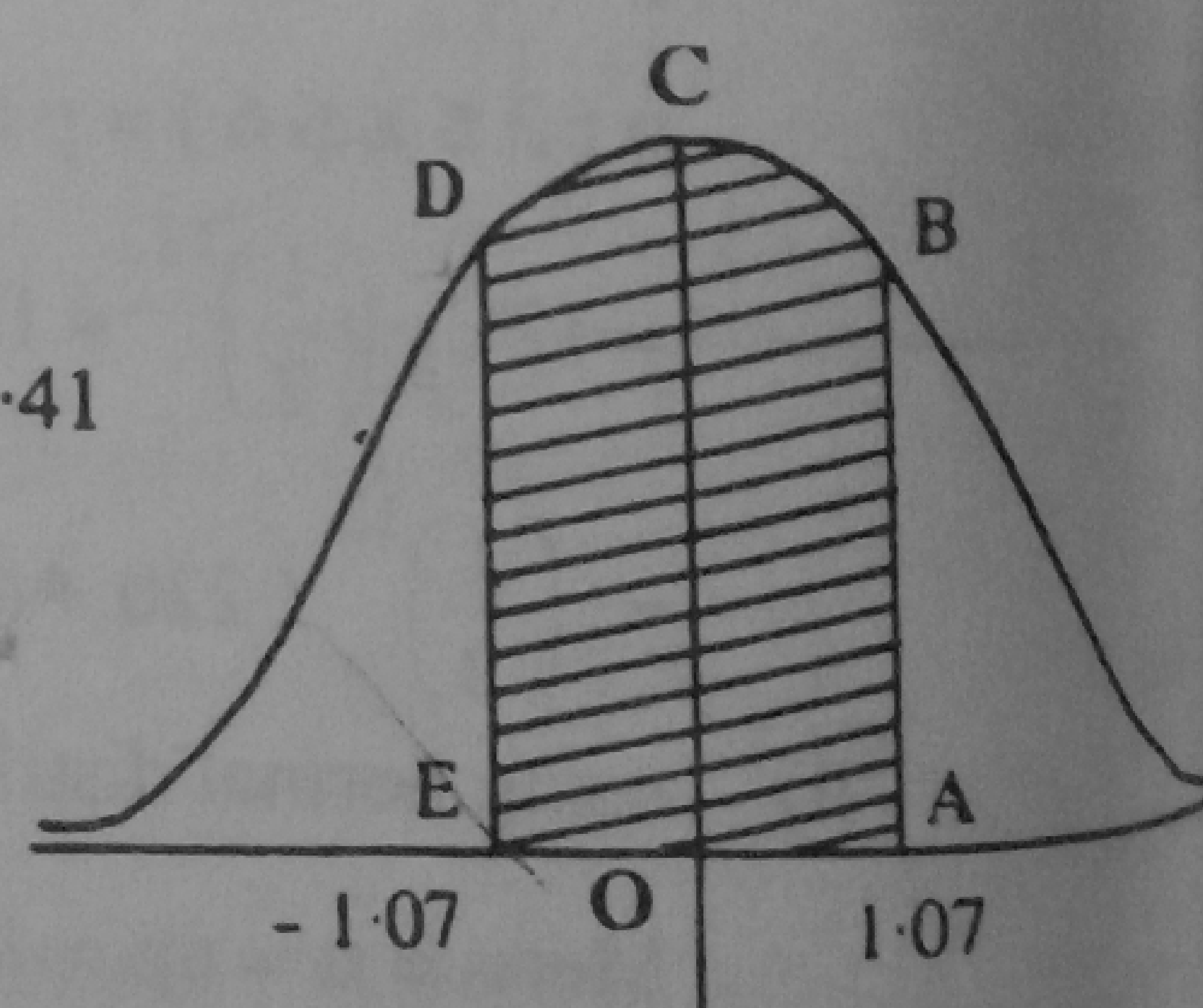


fig. 3.31

Probability $(4{\cdot}5 < x < 7{\cdot}5)$

$$= p(-1{\cdot}07 < z < 1{\cdot}07)$$

$$= \text{Area EABCDE} = \text{Area OABCO} + \text{Area EOCDE}$$

$$= p(0 \le z \le 1{\cdot}07) + p(-1{\cdot}07 \le z \le 0)$$

$$= p(0 \le z \le 1{\cdot}07) + p(0 \le z \le 1{\cdot}07)$$

$$= 2p(0 \le z \le 1{\cdot}07) = 2 \times {\cdot}3577 = {\cdot}7154$$

PROBLEM 3. ***Ten cards are drawn from a deck of 52 cards. Find the probability of getting 2 to 5 diamonds using normal distribution.***

Solution : p = probability of getting a diamond from 52 cards $= \frac{13}{52} = \frac{1}{4}$, n = 10

$$\mu = np = 10 \cdot \frac{1}{4} = 2{\cdot}5$$

$$\sqrt{npq} = \sqrt{10 \cdot \frac{1}{4} \cdot \frac{3}{4}} = 1{\cdot}37$$

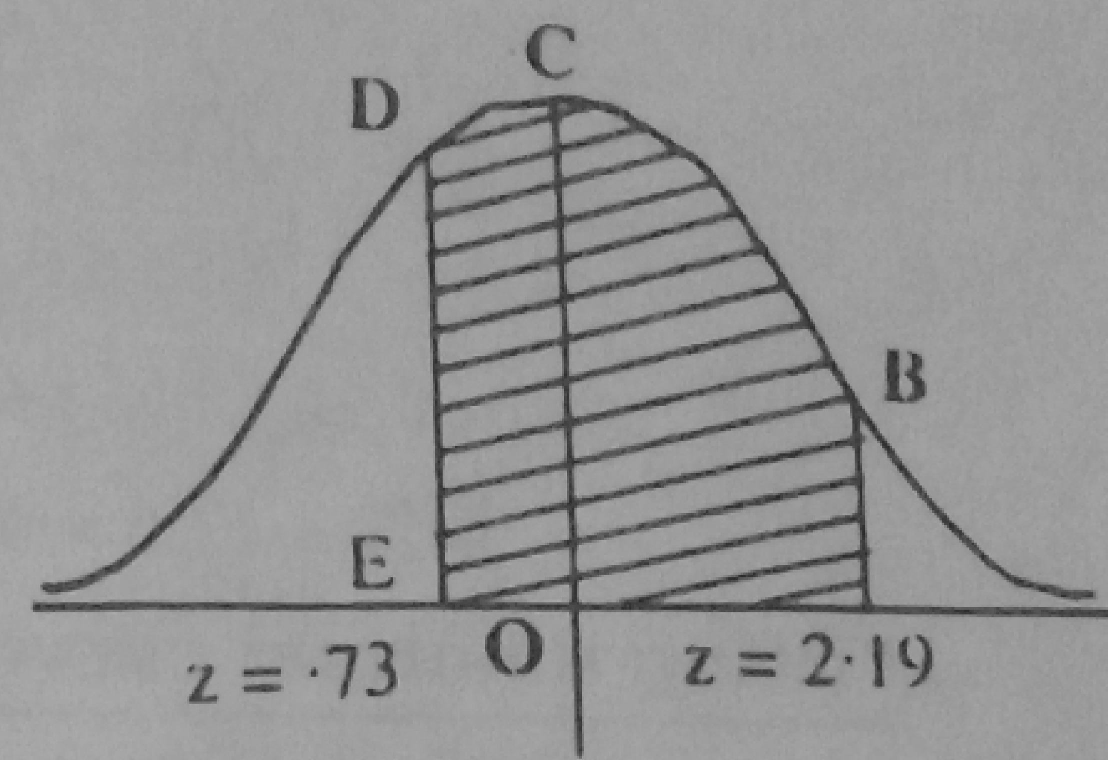


fig. 3.32

$$z_1 = \frac{2 - \frac{1}{2} - \mu}{\sigma} = \frac{1{\cdot}5 - 2{\cdot}5}{1{\cdot}37} = -{\cdot}73$$

$$z_2 = \frac{5 + \frac{1}{2} - \mu}{\sigma} = \frac{5{\cdot}5 - 2{\cdot}5}{1{\cdot}37} = 2{\cdot}19$$

$$p(z_1 \le z \le z_2) = p(-{\cdot}73 \le z \le 2{\cdot}19)$$

$$\begin{aligned} \text{Area EABCDE} &= p(\text{OABCO}) + p(\text{OCDEO}) \\ &= p(-{\cdot}73 \le z \le 0) + (0 \le z \le 2{\cdot}19) \\ &= p(0 \le z \le {\cdot}73) + p(0 \le z \le 2{\cdot}19) \\ &= {\cdot}2673 + {\cdot}4857 = {\cdot}7530 \end{aligned}$$

PROBLEM 4. ***Find the probability of getting an even number on face 3 to 5 times in throwing 10 dice. using***
(i) binomial distribution
(ii) normal distribution.

Solution : p = probability of getting even number on face in throwing a die $= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

n = 10 = no. of trials

$$p(3 \le x \le 5) = p(x = 3) + p(x = 4) + p(x = 5)$$

$$= 10C_3\left(\frac{1}{2}\right)^{10} + 10C_4\left(\frac{1}{2}\right)^{10} + 10C_5\left(\frac{1}{2}\right)^{10} = (120 + 210 + 252)\cdot\left(\frac{1}{2^{10}}\right) = \frac{582}{1024}$$

$$= {\cdot}568\dot{3}$$

ii) $\mu = np = 10 \times \frac{1}{2} = 5$

$$q = \sqrt{\frac{10}{4}} = 1{\cdot}58$$

$z_1 = \dfrac{2 \cdot 5 - 5}{1 \cdot 58} = -1 \cdot 58$

$z_2 = \dfrac{5 \cdot 5 - 5}{1 \cdot 58} = \cdot 32$

$p(z_1 \le z \le z_2)$ = Area EABCDE

= Area EOCDE + Area OABCO

$= p(-1 \cdot 58 \le z \le 0) + p(0 \le z \le \cdot 32)$

$= p(0 \le z \le 1 \cdot 58) + p(0 \le z \le \cdot 32)$

$= \cdot 4429 + \cdot 1255 = \cdot 5684$

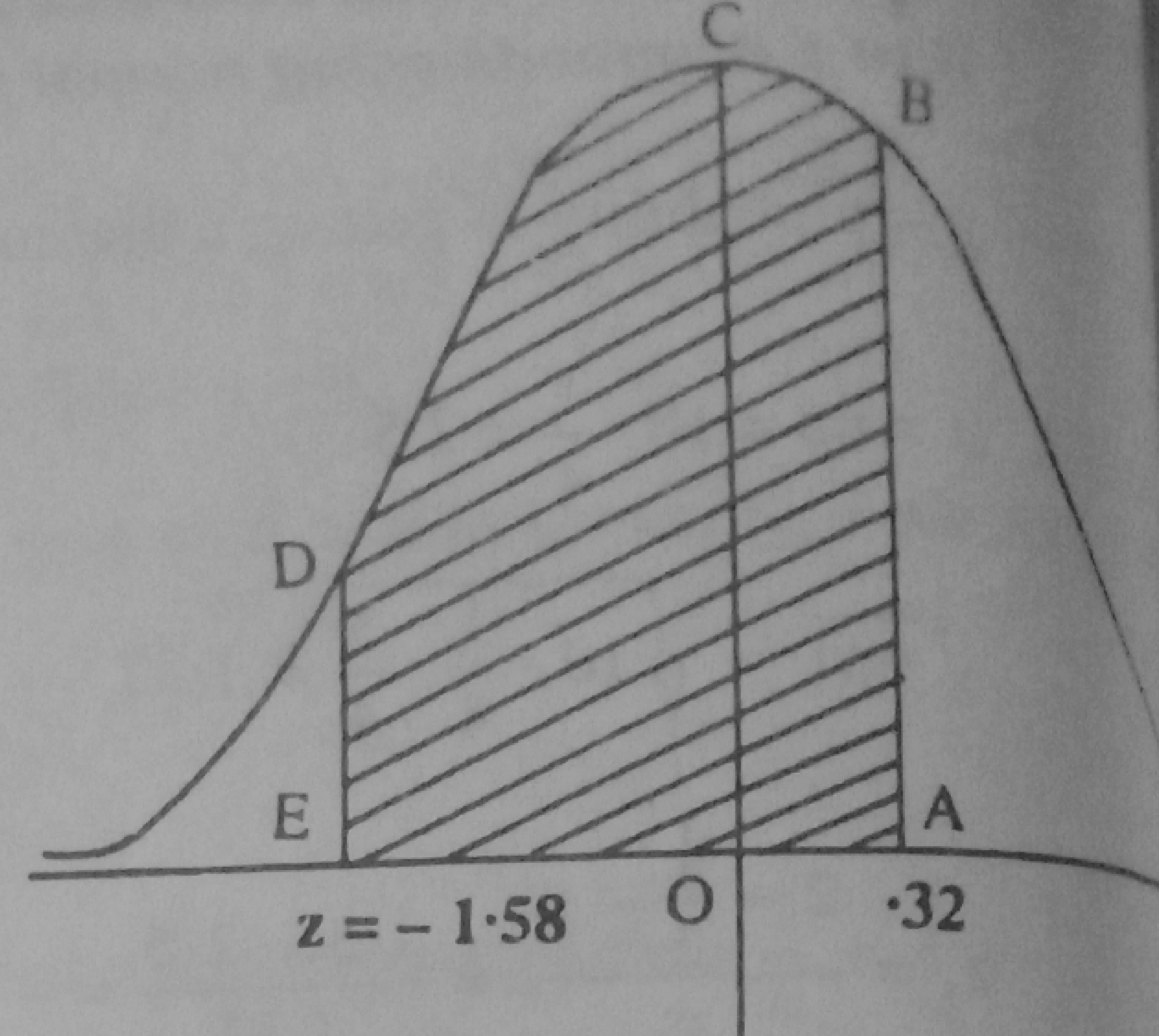


fig. 3.33

Fitting a normal curve

To find Expected frequencies :

Area Method :

1) If an interval lies between x_1 and x_2. Find the values of standard normal variates z_1 and z_2 by $z_1 = \dfrac{x_1 - \mu}{\sigma}$, $z_2 = \dfrac{x_2 - \mu}{\sigma}$.

2) From the table of area under the normal curve, find the area between $z = z_1$ and $z = z_2$.

3) Multiplying the area by N where $N = \sum_{i=1}^{n} f_i$ we get the expected frequencies.

◆ WORKED OUT PROBLEMS ◆

PROBLEM 1. ***Fit a normal curve and find the expected frequencies.***

Class	7·5 – 10·5	10·5 – 13·5	13·5 – 16·5	16·5 – 19·5	19·5 – 22·5	22·5 – 25·5	25·5 –
Frequency	12	10	15	19	12	14	8

Solution : $N = 90 \quad \mu = \text{Mean} = 18 \cdot 13, \sigma = 6 \cdot 28$

Class Interval	f	z_1 z_2	Area	Expected Frequency
7·5 – 10·5	12	– 1·69 – (– 1·21)	·0696	6·3
10·5 – 13·5	10	– 1·21 – (– ·74)	·1145	10·3
13·5 – 16·5	15	– ·74 – (– ·26)	·1678	15·1
16·5 – 19·5	19	– ·26 – ·22	·1897	17·07
19·5 – 22·5	12	·22 – ·7	·1709	15·4
22·5 – 25·5	14	·7 – 1·16	·119	10·7
25·5 – 28·5	8	1·16 – 1·65	·0735	6·6

2020.07.14 21:44

PROBLEM 2. *Fit a normal curve and find the expected frequencies*

x	8·60	8·59	8·58	8·57	8·56	8·55	8·54	8·53	8·52
f	2	3	4	9	10	8	4	1	1

Solution : Mean $\mu = 8{\cdot}56$, $\sigma = {\cdot}0177$, $N = 42$

Mid x	Class Interval	f	(z_1, z_2)	Area	Expected Frequency
8·60	8·605 – 8·595	2	2·54 – 1·97	·0189	·79
8·59	8·595 – 8·585	3	1·97 – 1·42	·0534	2·24
8·58	8·585 – 8·575	4	1·42 – ·8	·1341	5·63
8·57	8·575 – 8·565	9	·8 – ·28	·1778	7·5
8·56	8·565 – 8·555	10	·28 – (– ·28)	·2202	9·5
8·55	8·555 – 8·545	8	– ·28 – (– ·8)	·1778	7·5
8·54	8·545 – 8·535	4	– ·8 – (– 1·42)	·1341	5·63
8·53	8·535 – 8·525	1	– 1·42 – (– 1·97)	·0534	2·24
8·52	8·525 – 8·515	1	– 1·97 – (– 2·54)	·0189	·79

EXERCISE

1. Five cards are drawn from a pack of 52 cards. Find the probability i) there are 2 spades ii) $\geq$ 3 spades iii) None is a spade $\left(\frac{270}{4^5}, \left(1 - 34 \cdot \frac{3^5}{4^5}\right)\right)$

2. In a family of four children each find the probability that i) all are boys ii) only one boy iii) 2 boys $\left(\frac{1}{16}, \frac{1}{4}, \frac{3}{8}\right)$

3. In the above Problem find the number of families having i) all are boys ii) only one boy iii) 2 boys if there are 1000 families in a village **(63, 250, 375)**