DESIGN AND ANALYSIS OF ALGORITHMS (DAA) (A34EC)

By :-

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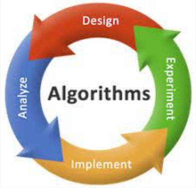

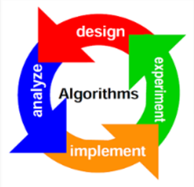

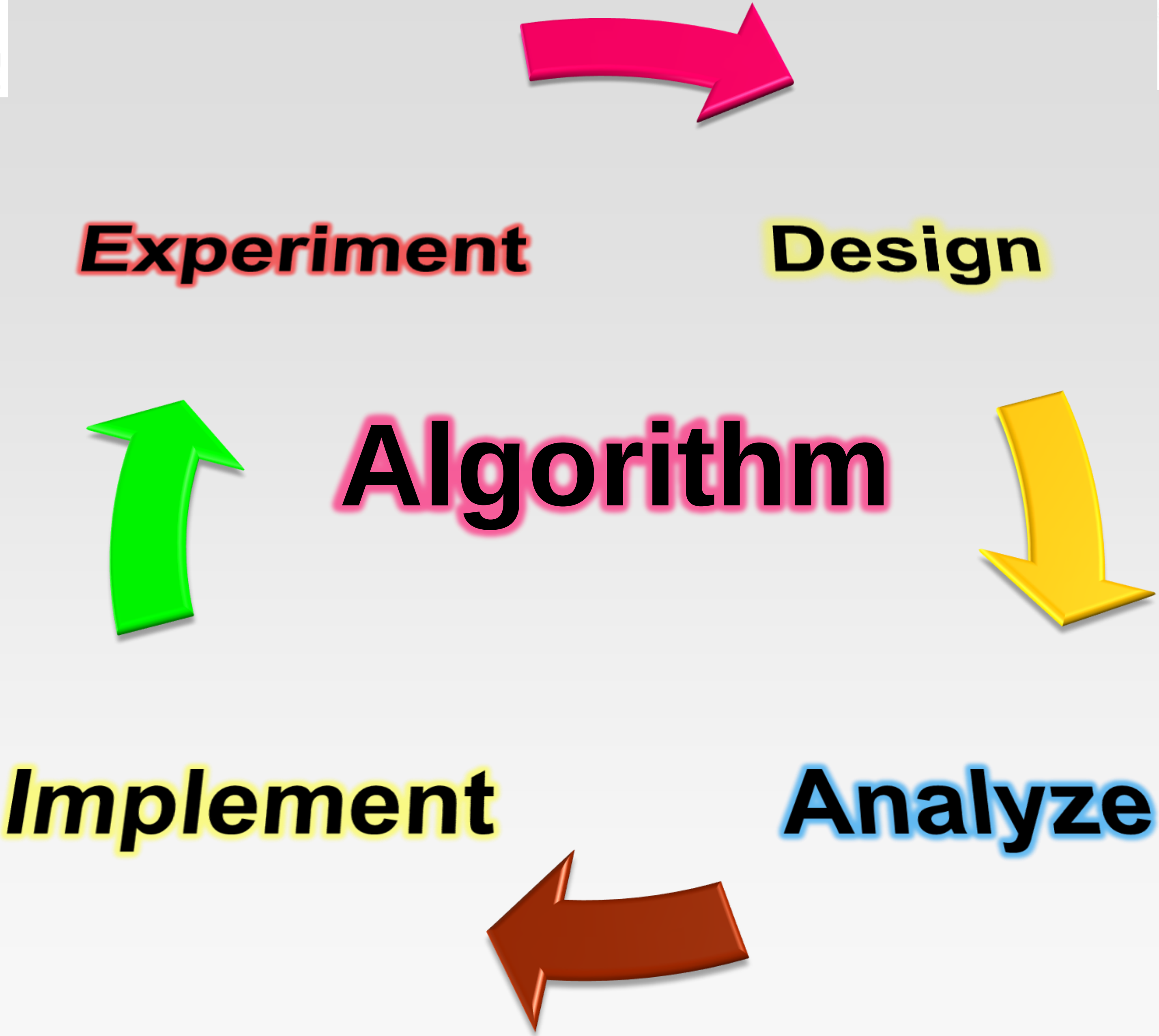
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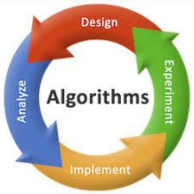
Algorithm
Implement
Analyze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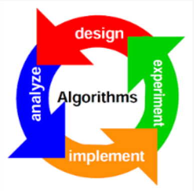
design
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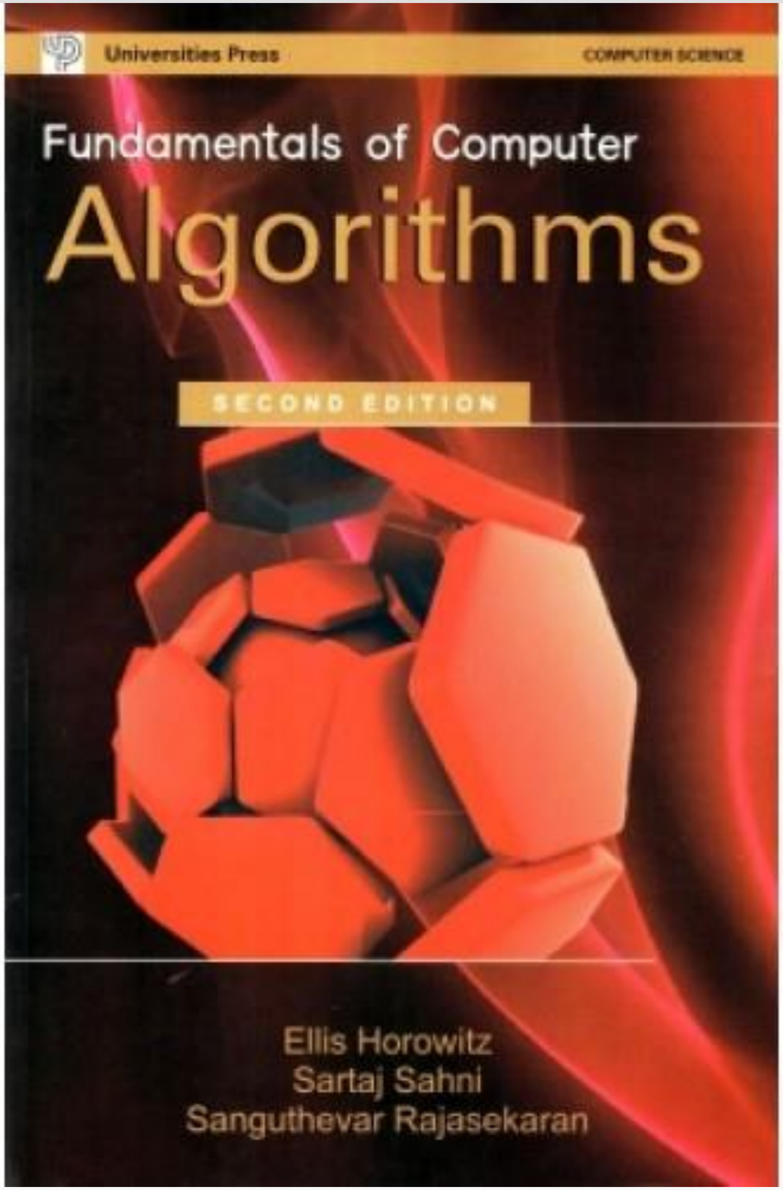
Textbook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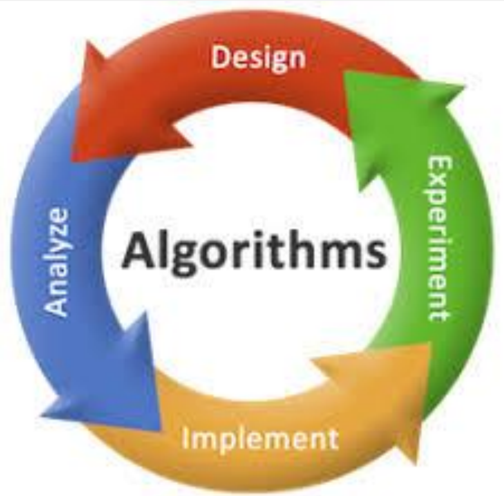
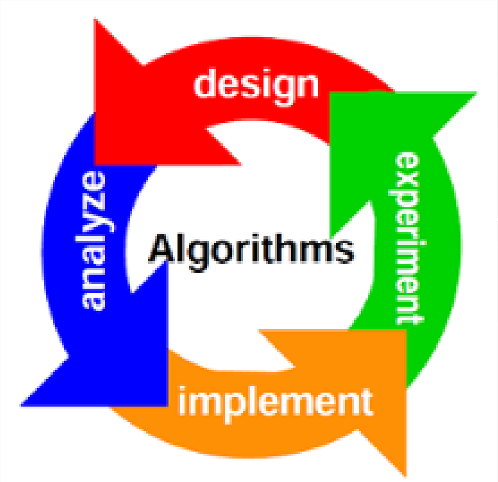
DAA Unit II

Disjoint Sets Divide and Conquer





Unit II Syllabus

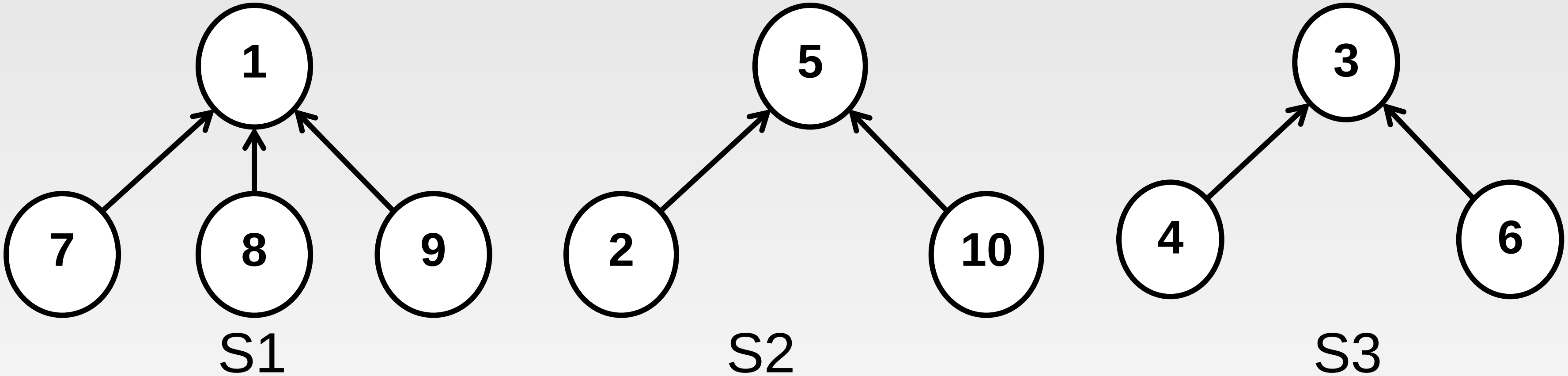
- Disjoint Sets:
  - Disjoint Sets,
  - Disjoint Set Operations,
  - Union and Find Algorithms,
  - Connected Components
  - and Bi-Connected Components.
- Divide and Conquer:
  - General method,
  - Applications Binary Search,
  - Merge Sort,
  - Strassen’s Matrix Multiplication.

Set and Disjoint Set

- A set is a collection data structure that stores certain values in a way that values are not repeated.
- Depending on whether these values are stored in an order or not, set is called ordered set or unordered set.
- It is an implementation of mathematical concept of Finite set.
- In this section we are going to see use of forests in the representation of sets.
- We assume that the elements of the sets are the numbers like 1, 2, 3, …., n.
- Also we assume that the sets being represented are pairwise **Disjoint** – that is if $\mathbf{S_i}$ **&** $\mathbf{S_j}$, $i \neq j$, are two sets then there is no element that is in both $\mathbf{S_i}$ **&** $\mathbf{S_j}$.

Set and Disjoint Set

- A disjoint-set data structure is a data structure that keeps track of a set of elements partitioned into a number of disjoint (non-overlapping) subsets.
- For Example consider n=10, the elements can be partitioned into three different sets S1, S2, S3 like…



Possible tree representation of Sets

- Note that for each set we have linked the nodes from children to parent.

Set and Disjoint Set

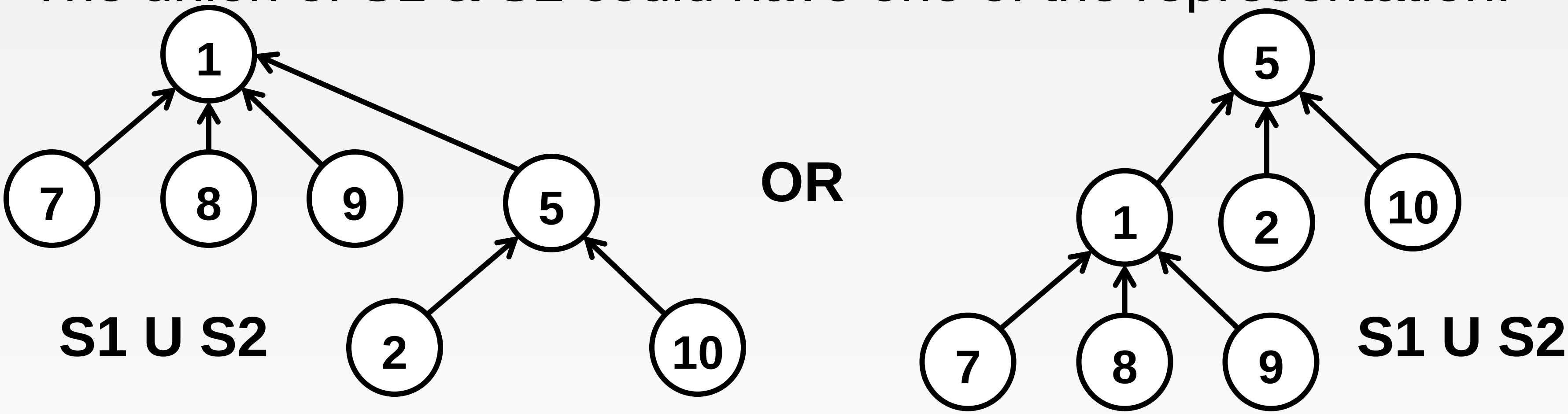
- The operations we wish to perform on these sets are :

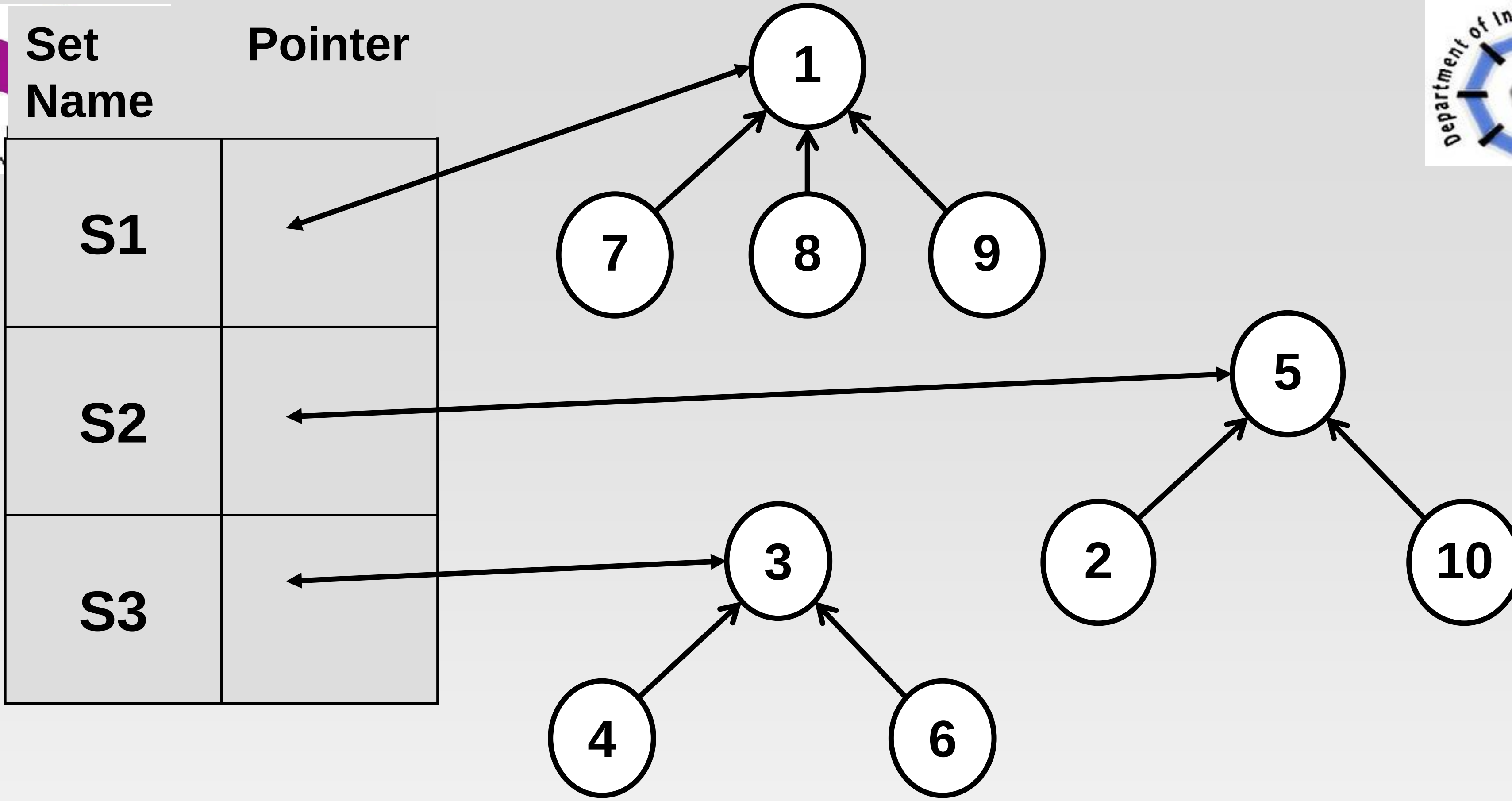
1. **Disjoint Set Union :** Join two subsets into a single subset. If $\mathbf{S_i}$ **&** $\mathbf{S_j}$ are two disjoint sets, then their union

   $\mathbf{S_i}$ **U** $\mathbf{S_j}$ = all elements x such that x is in $\mathbf{S_i}$ or $\mathbf{S_j}$.

2. **Find(i) :** Given the element i, find the set containing i. Determine which subset a particular element is in. This can be used for determining if two elements are in the same subset.

- **Union and Find O**perations : Consider Union operation first The union of S1 & S2 could have one of the representation.





Data representation for S1, S2 & S3

i	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
p[i]	-1	5	-1	3	-1	3	1	1	1	5

Array representation for S1, S2 & S3

Union and Find Algorithms

- Here are the algorithm for Union & Find operations.

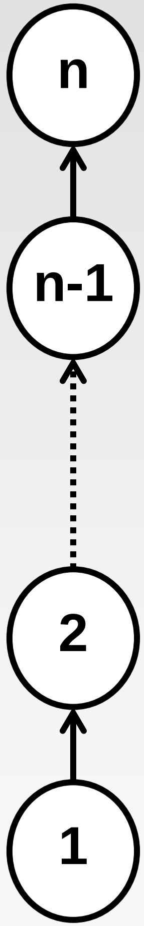
1. **Algorithm SimpleUnion (i, j)**
2. **{**
3. **p [i] := j;**
4. **}**

1. **Algorithm SimpleFind (i)**
2. **{**
3. **while (p [i] >=0) do**
4. **i := p [i];**
5. **return i;**
6. **}**

- Even these algorithms are very easy to state, their performance are not very good.

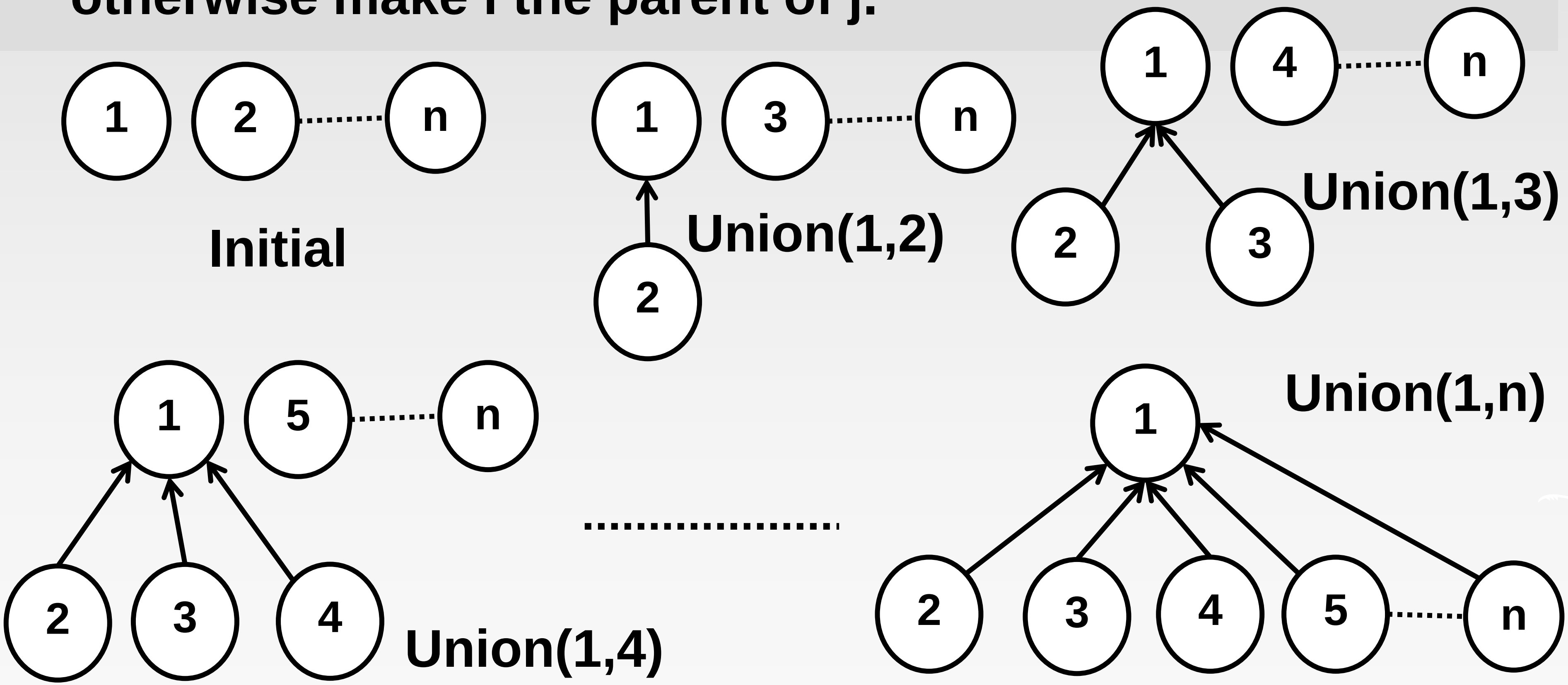
Union and Find Algorithms

- For example if we start with **q** elements each in set of its own i.e. $\mathbf{S_i = \{ i \}, 1 \le i \le q}$ then initial configuration consists of a forest with q nodes and $\mathbf{p[\,i\,] = 0, 1 \le i \le q}$
- Now let us process following sequence of union-find operations
  - **Union(1,2), Union(2,3), Union(3,4), .., Union(n-1, n)**
  - **Find(1), Find(2), Find(3), Find(4), ….., Find(n)**
- This will results in the degenerate tree.
- Since time taken for union is constant, the **n-1** union can be processed in time **O(n)**.
- Since the time required to process a find for an element at level i of a tree is **O(i)**, the total time needed to process n finds is $\mathbf{O(\sum_{i=1}^{n} i) = O(n^2)}$



Union and Find Algorithms

- We can improve performance of Union & Find algorithms using weighting rule for **Union(i, j)**.
- If the number of nodes in the tree with **root i** is **less than** the number in the tree with **root j**, then make **j the parent of i; otherwise make i the parent of j.**



Union and Find Algorithms

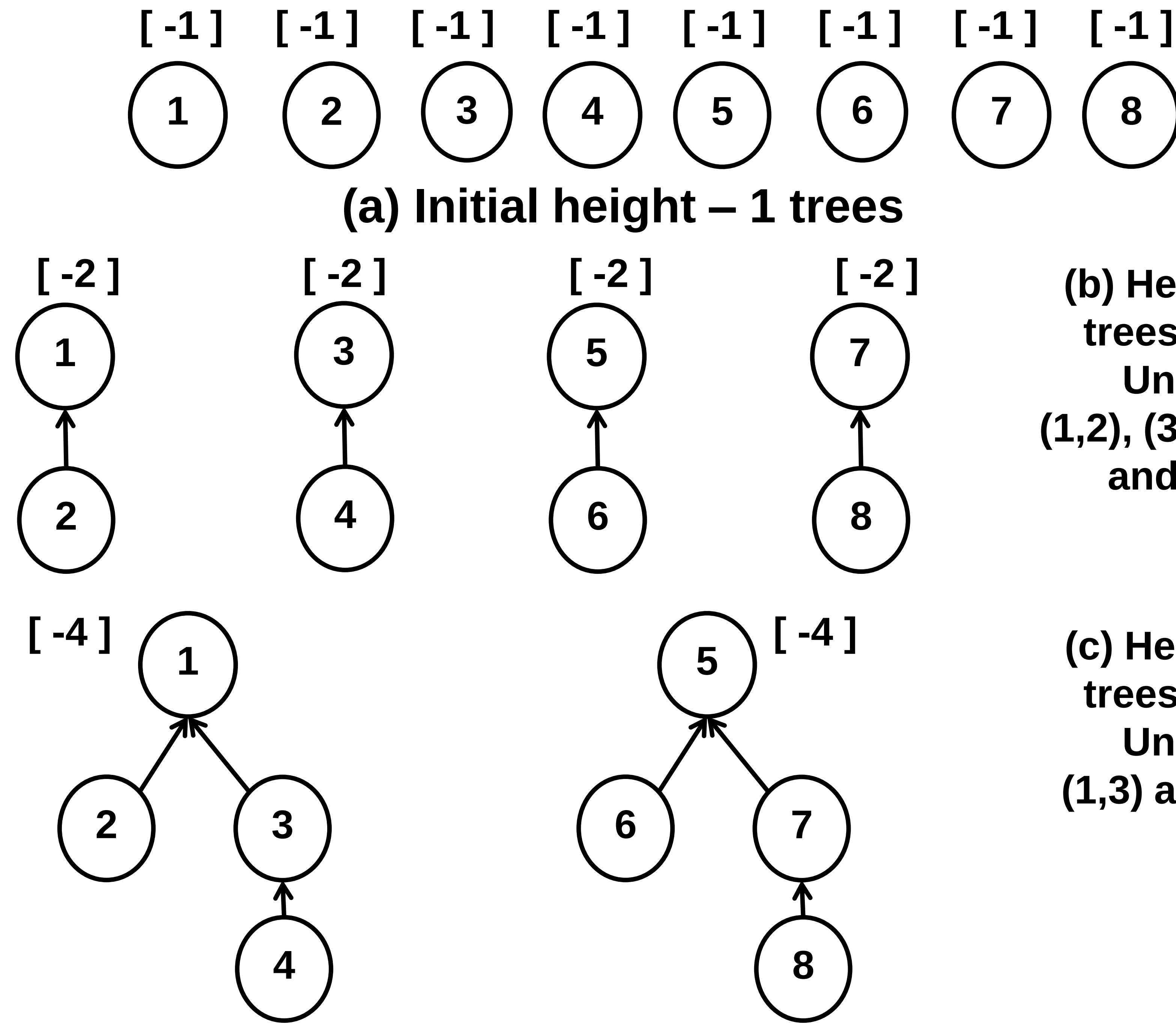
```
Algorithm WeightedUnion (i, j)
// p [ i ] = - count[ i ] and p [ j ] = - count[ j ]
{
  temp := p [ i ] + p [ j ];
   if(p [ i ] > p [ j ]) then
   { // i  has fewer nodes
      p [ i ] := j; p [ j ] := temp;
   }
   else
    { // j  has fewer or equal nodes
      p [ j ] := i; p [ i ] := temp;
   }
 }
```

Union and Find Algorithms

- The Find algorithm remains unchanged.
- The maximum time to perform a find will be calculated as
- Assume that we start with a forest of trees, each having one node. Let T be a tree with m nodes created using WeightedUnion. The height of T is no greater than $\mathbf{\log_2 m + 1}$
- Now consider creation of tree using WeightedUnion algorithm for **8** elements with initial configuration

$$p[\, i \,] = -\, count[\, i \,] = -1,\ 1 \le i \le 8 = n$$

  - **Union(1,2), Union(3,4), Union(5,6), Union(7,8)**
  - **Union(1,3), Union(5,7), and finally Union(1,5)**
- This will results in tree as in next slides.
- We found that height of each tree with m nodes is $\mathbf{\log_2 m + 1.}$
- The time to process a find is **O(log m)** if there are m elements
- If an intermixed sequence of **u-1** union and **f** find operations, the time becomes **O(u + f log u)** along with **O(n)** to initialize **n-tree** forest.

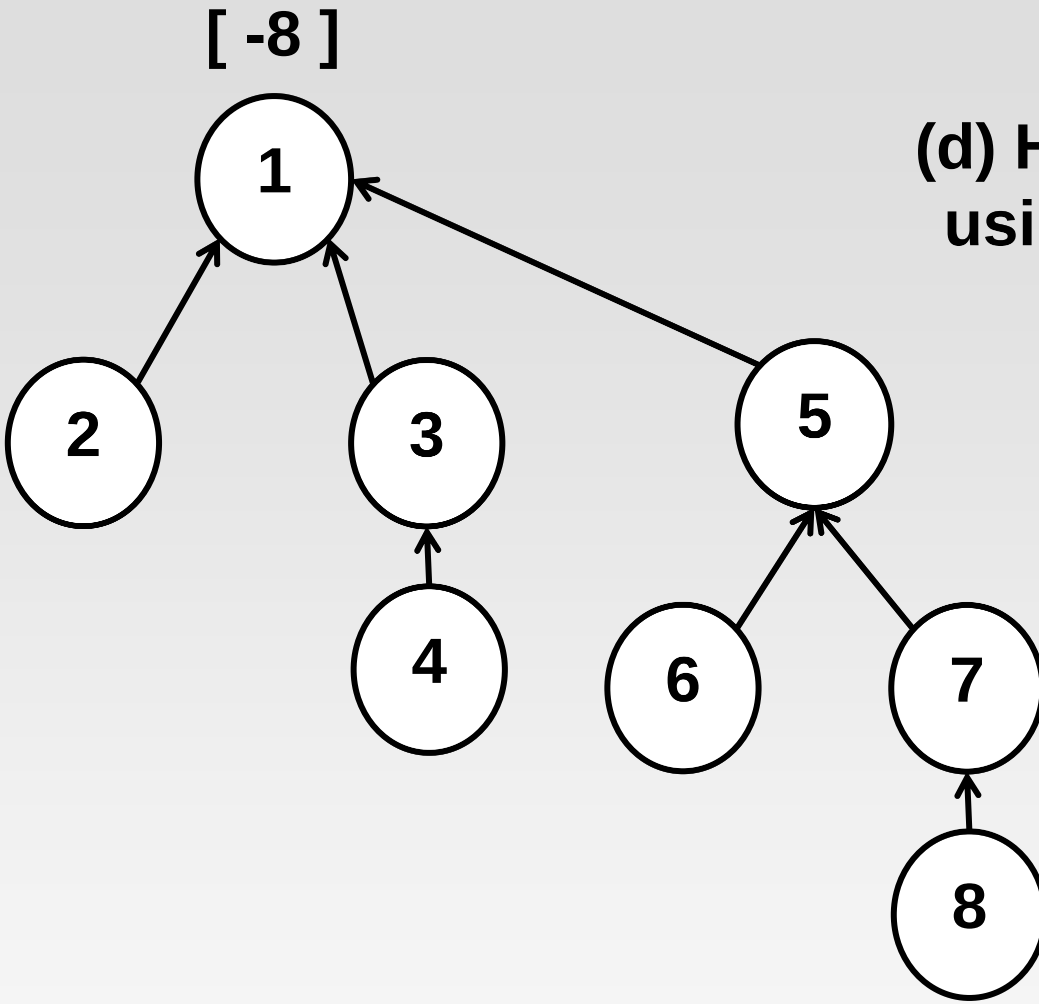


(a) Initial height – 1 trees

(b) Height – 2 trees using Unions (1,2), (3,4), (5,6), and (7,8)

(c) Height – 3 trees using Unions (1,3) and (5,7)

(d) Height – 4 trees using Union (1,5)



Trees achieving worst-case bound

Union and Find Algorithms

- Still further improvement is possible in the Find algorithm using Collapsing Rule.
- **Collapsing Rule :** If j is node on the path from i to its root and **p[i] ≠ root[i]**, then set **p[j] to root [i].**

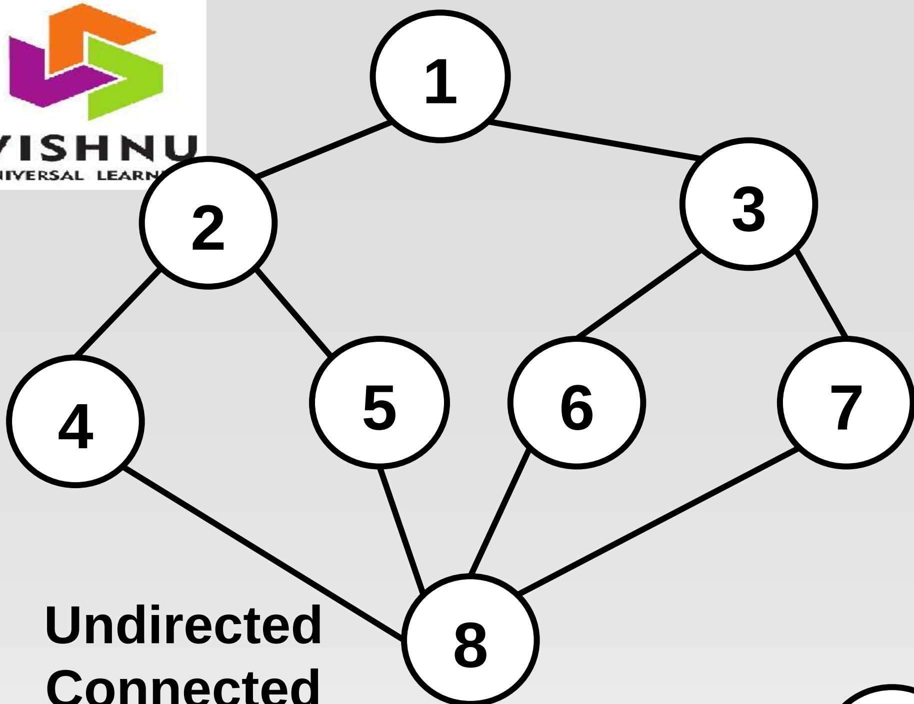
```
Algorithm CollapsingFind (i)
{
    r := i;
    while (p [ r ] > 0) do
        r := p [ r ];
    while ( i ≠ r ) do
    {
        s := p [ i ];
        p [ i ] := r;
        i := s;
    }
return r
}
```

Union and Find Algorithms

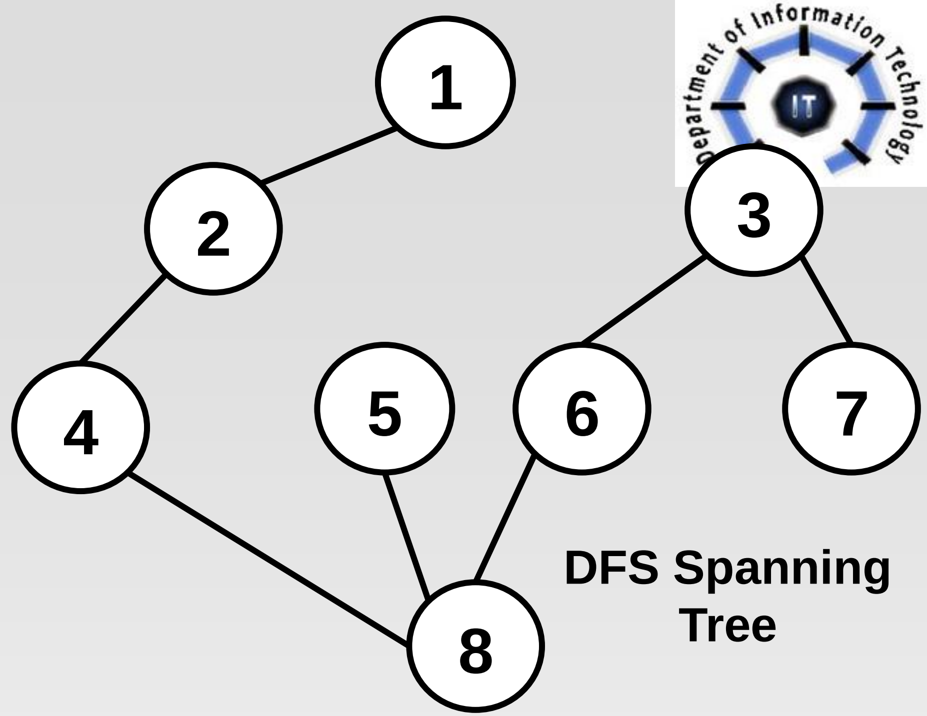
- Consider the tree created by WeightedUnion as seen in previous example. Now we will process the following ten finds to search element 8:
  - **Find(8), Find(8), Find(8), Find(8), ….., Find(8)**
- If we use **SimpleFind**, each Find(8) requires going up three parent link fields for a total of 30 moves to process ten finds.
- When we use **CollapsingFind** the first Find(8) requires going up **three** links and then resetting **two** links (Actually three links as it will reset parent of 5 to 1).
- Each remaining nine Find requires going up by only one link field.
- The total cost is now only **15** moves.

Connected Components

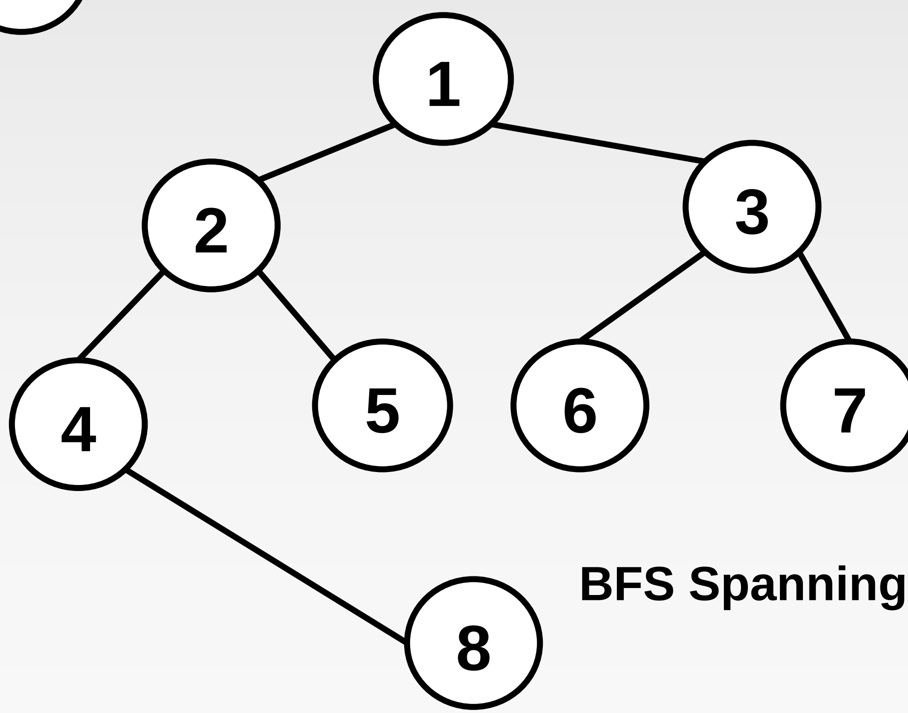
- **Graph** : A graph **G** consists of two sets **V** and **E** where **V** is set of vertices and **E** is a set of pairs of edges. **G=(V, E).**
- In an undirected graph **G**, two vertices **u** & **v** are said to be connected iff there is a path in **G** from **u** to **v**.
- An undirected graph is said to be connected iff for every pair of distinct vertices **u** & **v** in **V(G)**, there is a path from **u** to **v** in **G**.
- A connected component of an undirected graph is a maximal connected subgraph. Maximal meaning that **G** contains no other subgraph.
- If graph **G** is connected undirected graph, then all vertices of **G** will get visited on the first call to Breadth First Search (BFS). If not then connected, then at least two calls to BFS.



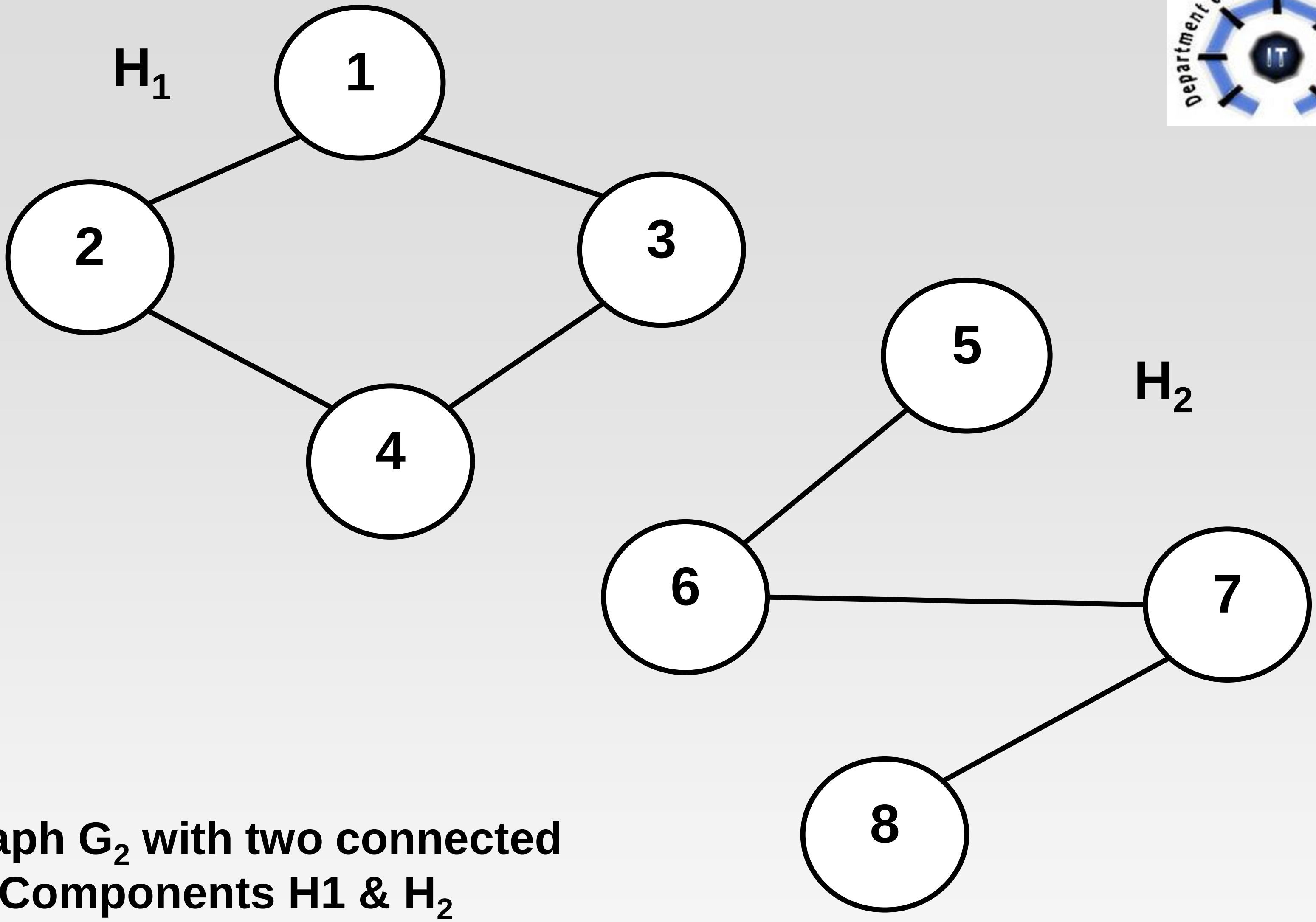
Undirected Connected Graph G_1



DFS Spanning Tree



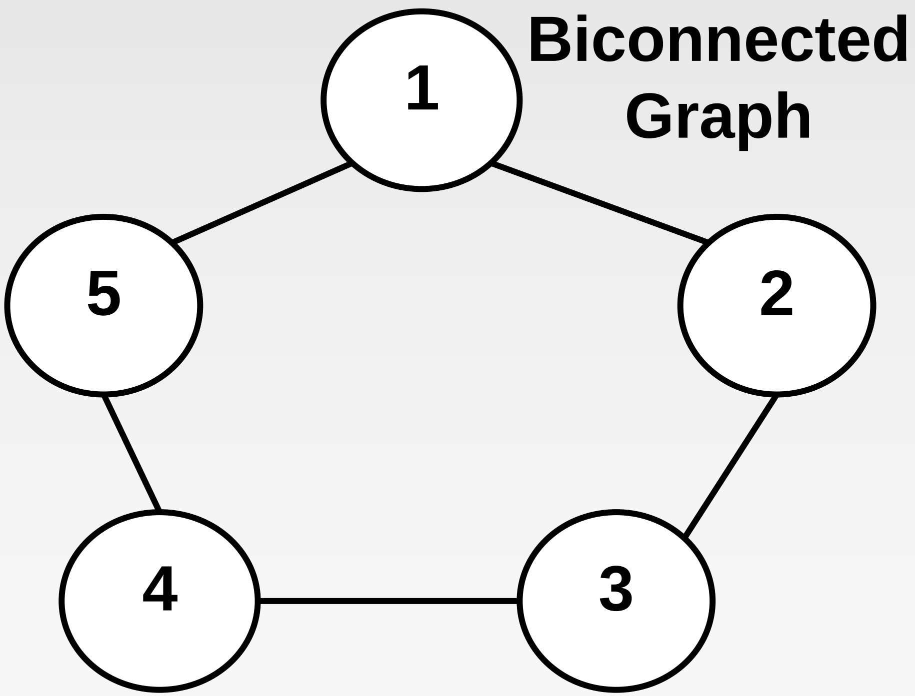
BFS Spanning Tree

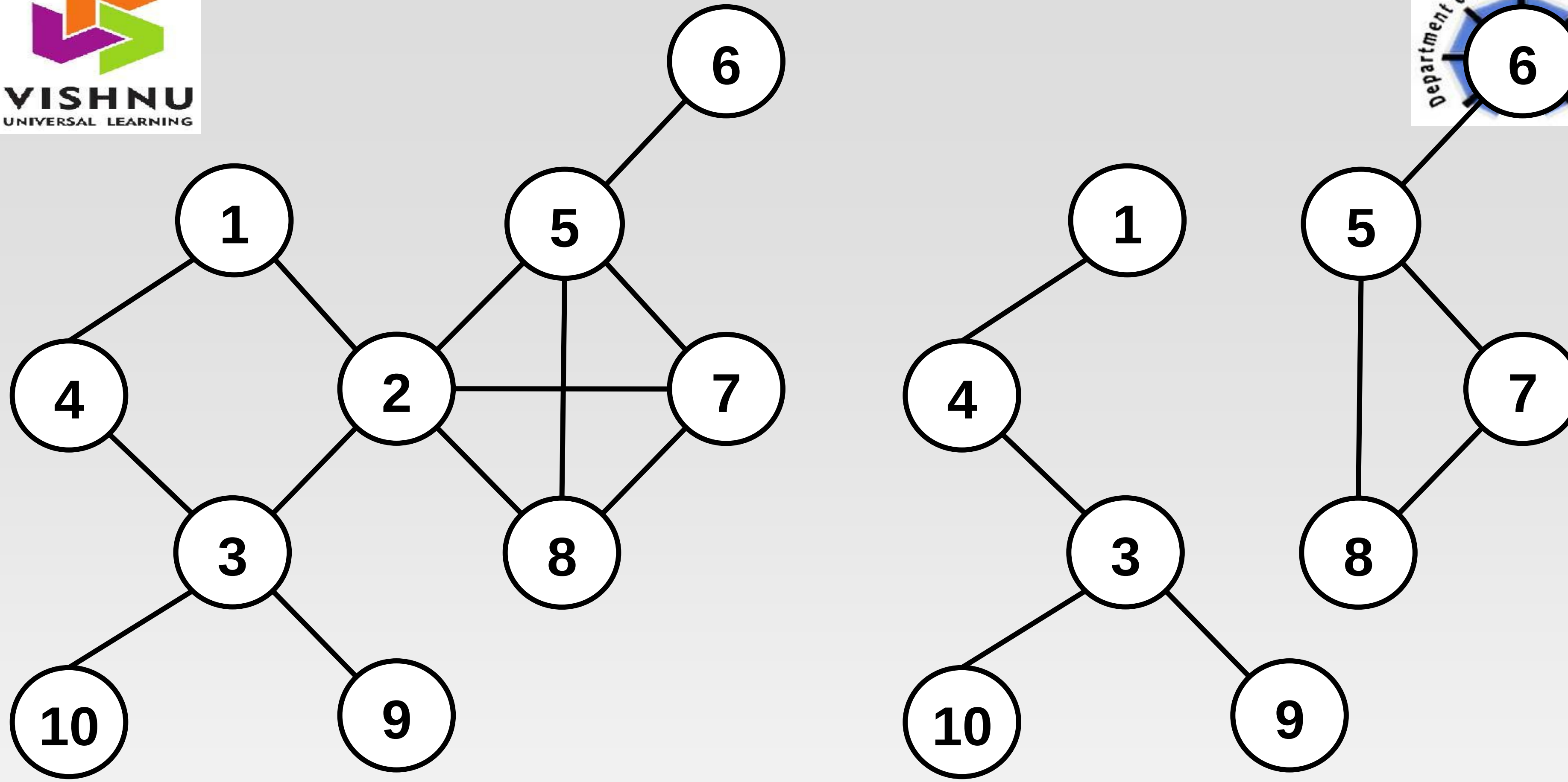


A Graph G_2 with two connected Components H1 & H_2

Biconnected Components

- **Articulation Point** : A vertex **v** in a connected graph **G** is an articulation point if and only if the deletion of vertex **v** together with all edges incident to **v** disconnects the graph into two or more nonempty components.
- **Biconnected** : A graph **G** is biconnected if and only if it contains no articulation points.
- The presence of articulation points in connected graph is undesirable feature.
- The graph shown in figure is biconnected graph example.
- The graph shown in next slide figure is not biconnected graph as deleting vertex 2 we will get two Graphs.



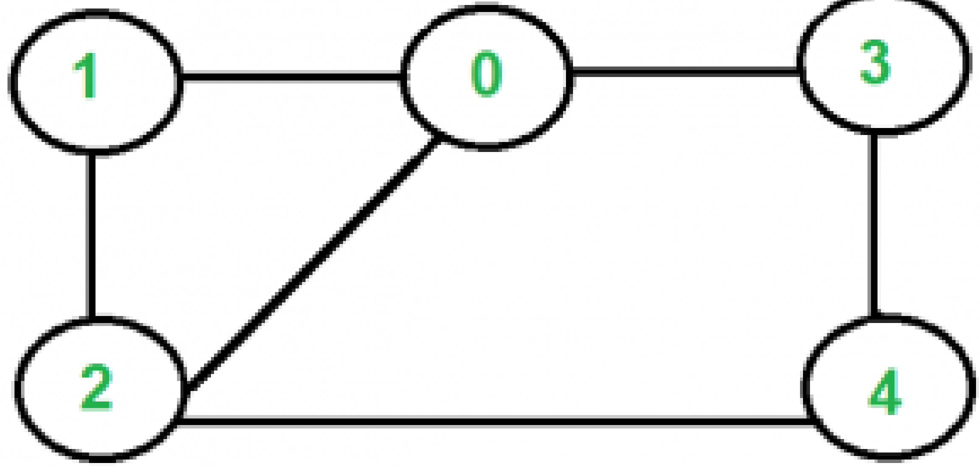


(a) Graph G

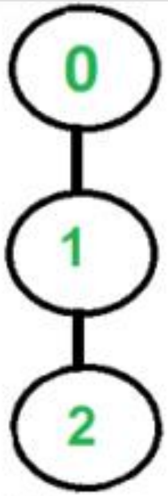
(b) Result of deleting vertex 2

An Example Graph G – Not Biconnected Graph

Biconnected

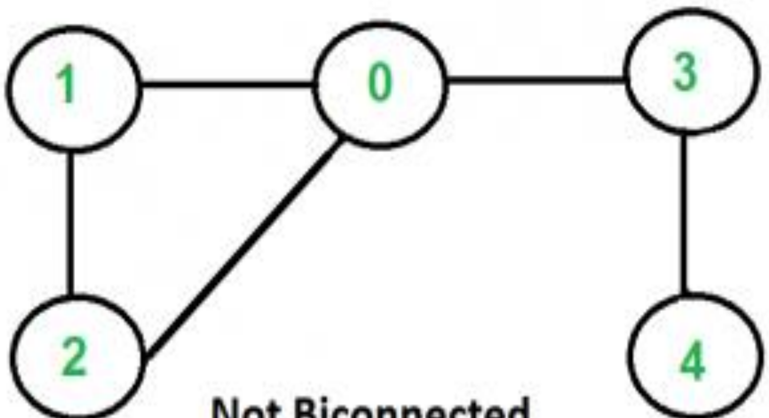


Biconnected

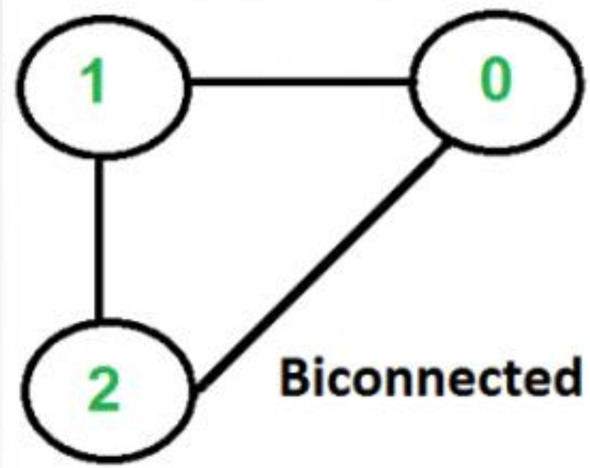


Not Biconnected

Few more Biconnected & Non Biconnected Graphs



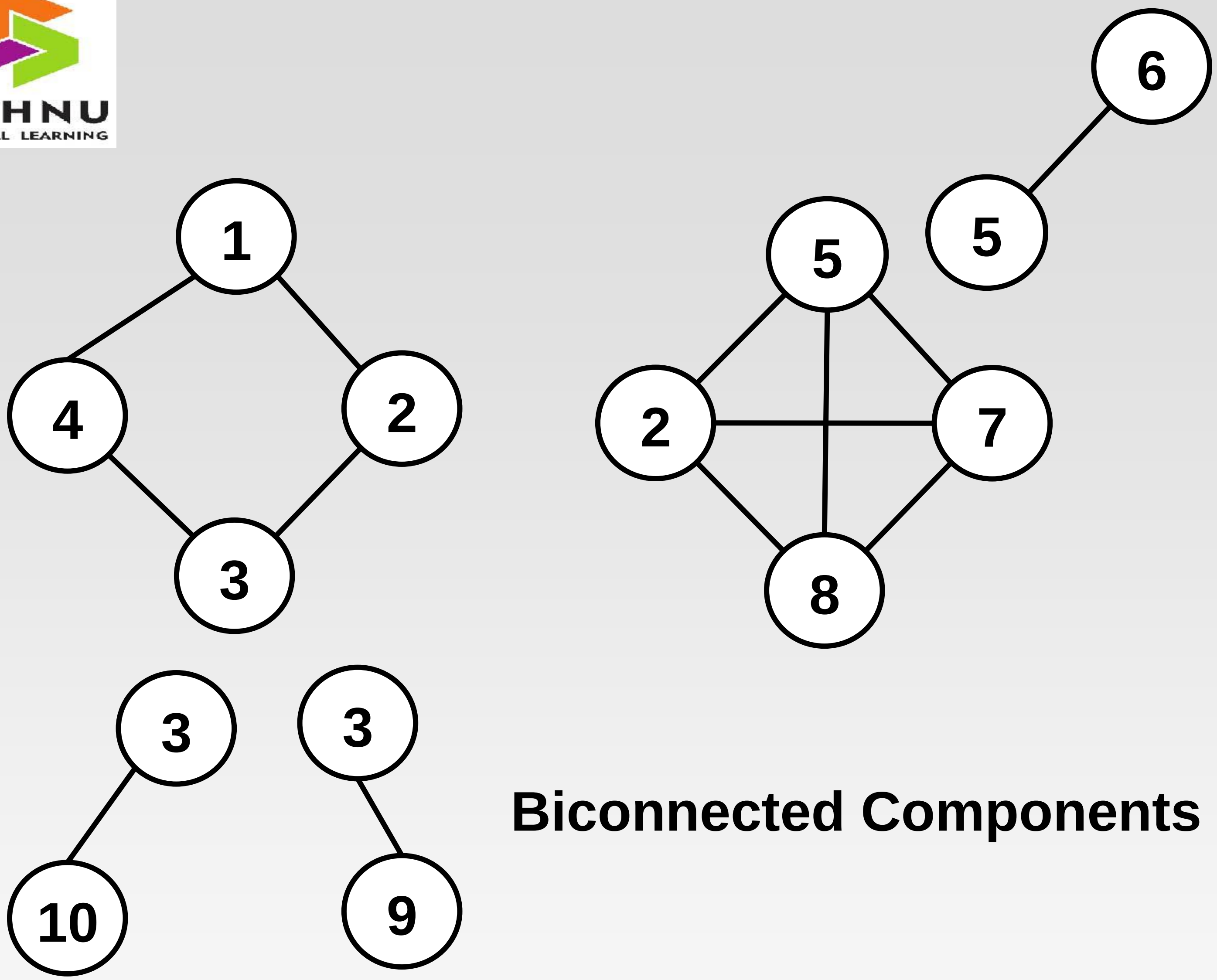
Not Biconnected



Biconnected

Biconnected Components

- For example, if **G** represents a communication network with the vertices representing communication stations and edges communication lines, then failure of a communication station **i** that is an articulation point would result in the loss of communication to points other than **i** too.
- On the other hand, if **G** has no articulation point, then if any station **i** fails, we can still communicate between every two stations not including station **i**.
- We will see an efficient algorithm to test whether a connected graph is biconnected.
- For the case of graphs that are not biconnected, this algorithm will identify all articulation points.
- Next slide shows the biconnected components of previous example Graph **G**.



Biconnected Components of Graph G

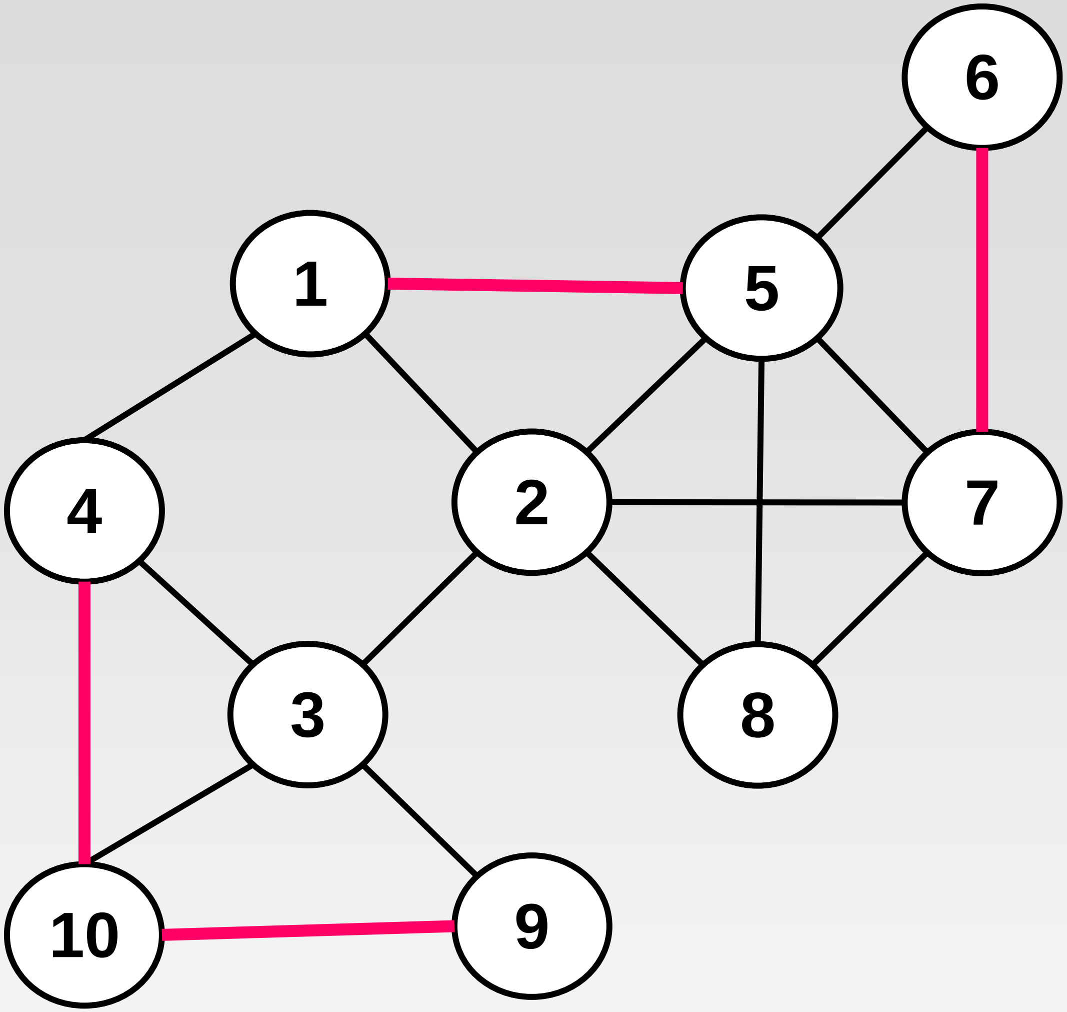
Biconnected Components

- It is easy to show that, Two biconnected component can have at most one vertex in common and this vertex as an articulation point.
- The graph **G** can be transformed into a biconnected graph by using edge addition scheme algorithm.

1. **for** each articulation point a **do**
2. **{**
3. Let $\mathbf{B_1, B_2, \ldots., B_k}$ be the biconnected
4. components containing vertex a;
5. Let $\mathbf{v_i, v_i \neq a,}$ be a vertex in $\mathbf{B_i, 1 \leq i \leq k;}$
6. Add to **G** the edges $\mathbf{(v_i, v_{i+1}), 1 \leq i < k;}$
7. **}**

Biconnected Components

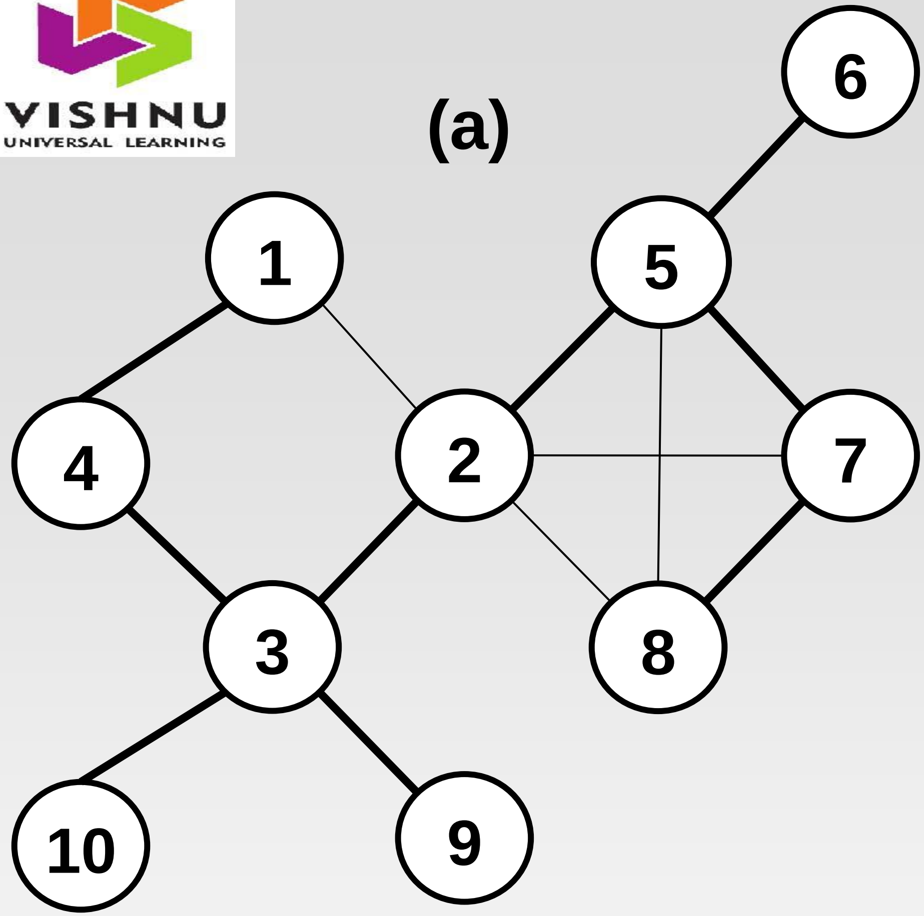
- Using the above scheme to transform the graph **G** seen in previous example into a biconnected graph requires us to…
- Add edges (4, 10) and (10, 9) corresponding to articulation point 3.
- Addedge (1, 5) corresponding to articulation point 2.
- And add edge (6, 7) corresponding to articulation point 5.
- Note that once the edges (v_i, v_{i+1}) as per line 6 (of algorithm in previous slide) are added, vertex **a** is no longer an articulation point.
- We can conclude that addition of the edges corresponding to all articulation points, G has no articulation point and so it is biconnected graph

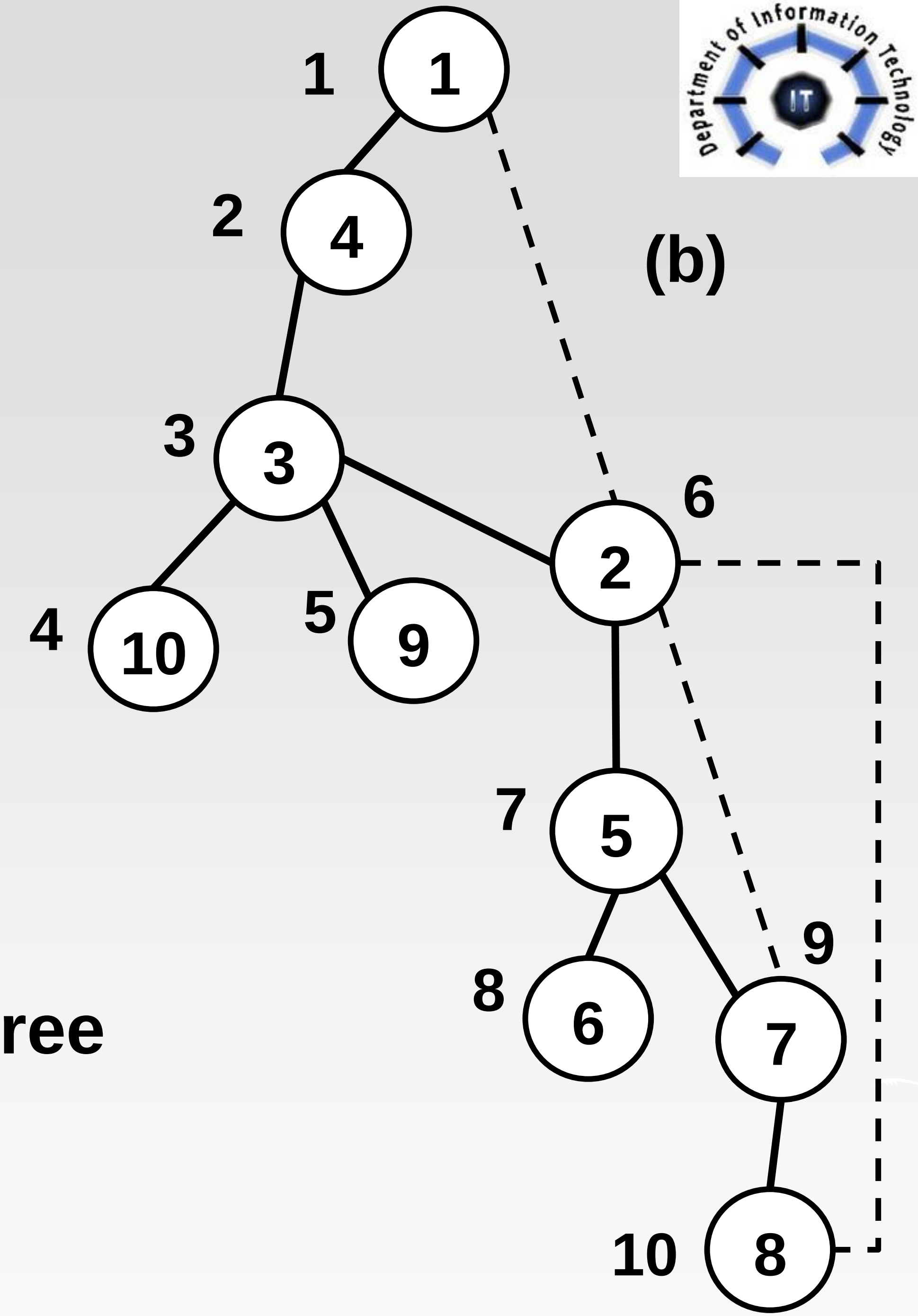


Biconnected graph corresponding to Graph G

Biconnected Components

- Now, we will see how to identify the articulation points and biconnected components of a connected graph **G** with **n >= 2** vertices.
- The problem is efficiently solved by using **Depth First Search (DFS)** Spanning Tree.
- The depth first spanning tree of the Graph **G** is shown in next slide.
- There is a number outside each vertex.
- These numbers correspond to the order in which a depth first search visits these vertices and are referred to as **depth first numbers (dfns)** of the vertex.
- The **solid edges** form the depth first spanning tree are called as **tree edges** and **broken edges** (all other edges) are called **back edges**.





A depth first spanning tree of the Graph G

Biconnected Components

- The **root node** of a depth first spanning tree is an **articulation point** iff it has **at least two children**
- Also if **u** is any other vertex, then it is not an articulation point iff from every child **w** of **u** it is possible to reach an ancestor of **u** using only a path made up of descendants of **w** and a back edge.
- Note that if this cannot be done for some child **w** of **u**, them the deletion of vertex **u** leaves behind at least two nonempty components, this observation leads to a simple rule to identify articulation points.
- For each vertex ***u***, define ***L***[***u***] as follows :

$$L[\,u\,] = \min \{\, dfn[\,u\,],\ \min \{\, L[\,w\,] \mid w \text{ is a child of } u \,\},\ \min \{\, dfn[\,w\,] \mid (u, w) \text{ is a back edge} \,\}\,\}$$

- If ***u*** is not the root, then ***u*** is an articulation point iff ***u*** has a child w such that $L[\,w\,] >= dfn[\,u\,]$

Biconnected Components

- **Example :** For spanning tree of Graph G - as shown in previous slide fig (b) the L values are

L [1:10] = { 1, 1, 1, 1, 6, 8, 6, 6, 5, 4 }

And dfn [1:10] = { 1, 6, 3, 2, 7, 8, 9, 10, 5, 4 }

- Vertex 3 is an articulation point as child 10 has

  L[10] = 4 and dfn[3] = 3

- Vertex 2 is an articulation point as child 5 has

  L[5] = 6 and dfn[2] = 6

- Vertex 5 is an articulation point as child 6 has

  L[6] = 8 and dfn[5] = 7

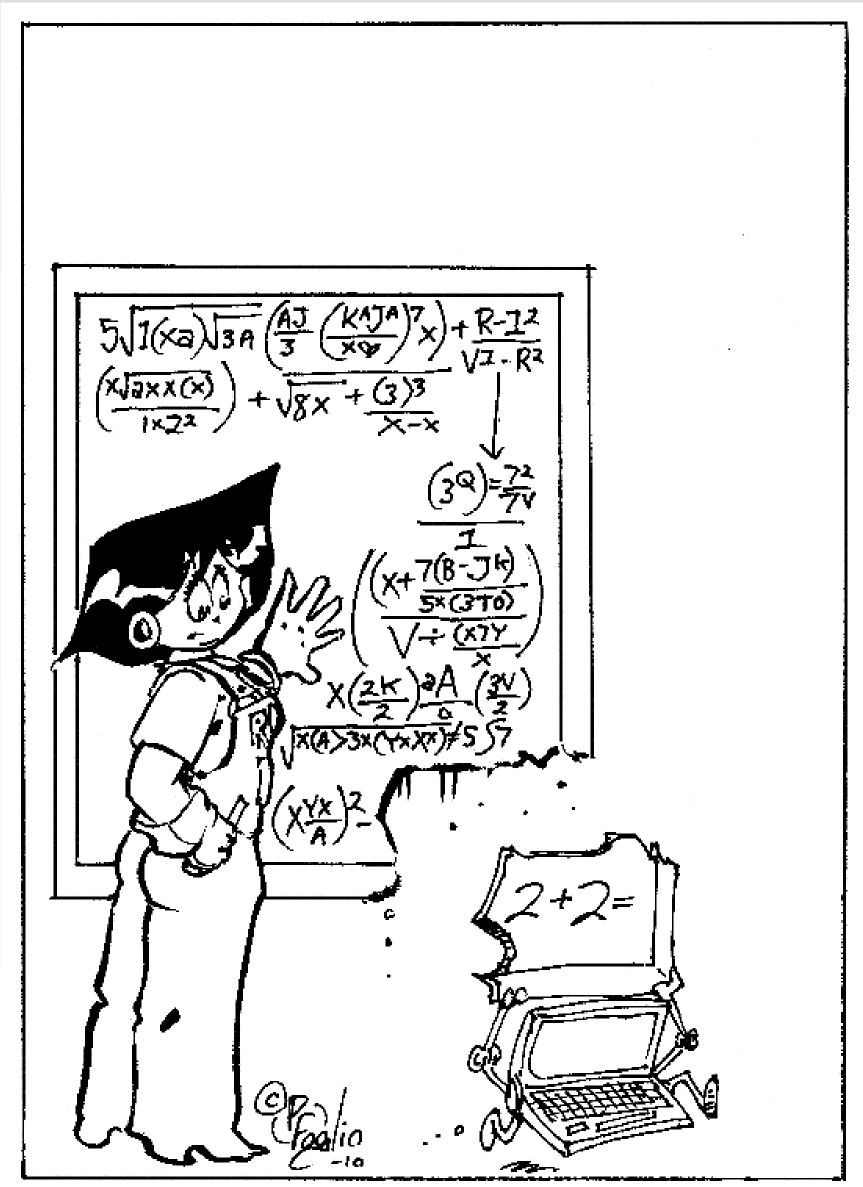
```
Algorithm Art(u, v)
// u is a start vertex for depth first search. v is its parent if any
// in the depth first spanning tree. It is assumed that the global
// array dfn is initialized to zero and that the global variable
// num is initialized to 1. n is the number of vertices in G.
{
    dfn[u] := num; L[u] := num; num := num + 1;
    for each vertex w adjacent from u do
    {
        if (dfn[w] = 0) then
        {
            Art(w, u); // w is unvisited.
            L[u] := min(L[u], L[w]);
        }
        else if (w ≠ v) then L[u] := min(L[u], dfn[w]);
    }
}
```

Algorithm 6.9 Pseudocode to compute *dfn* and *L*

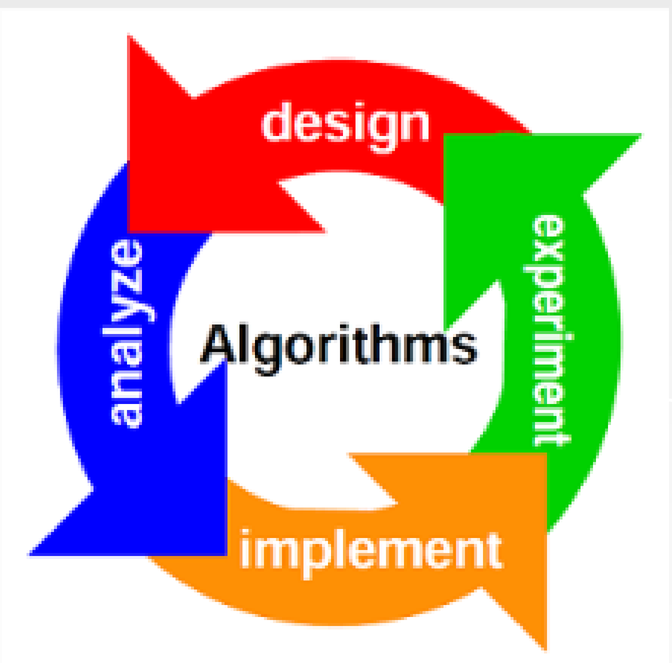
```
Algorithm BiComp(u, v)
// u is a start vertex for depth first search. v is its parent if
// any in the depth first spanning tree. It is assumed that the
// global array dfn is initially zero and that the global variable
// num is initialized to 1. n is the number of vertices in G.
{
    dfn[u] := num; L[u] := num; num := num + 1;
    for each vertex w adjacent from u do
    {
        if ((v ≠ w) and (dfn[w] < dfn[u]))then
            add (u, w) to the top of a stack s;
        if (dfn[w] = 0) then
        {
            if (L[w] ≥ dfn[u]) then
            {
                write ("New bicomponent");
                repeat
                {
                    Delete an edge from the top of stack s;
                    Let this edge be (x, y);
                    write (x, y);
                } until (((x, y) = (u, w)) or ((x, y) = (w, u)));
            }
            BiComp(w, u); // w is unvisited.
            L[u] := min(L[u], L[w]);
        }
        else if (w ≠ v) then L[u] := min(L[u], dfn[w]);
    }
}
```

Algorithm 6.10 Pseudocode to determine bicomponents

Divide-and-Conquer



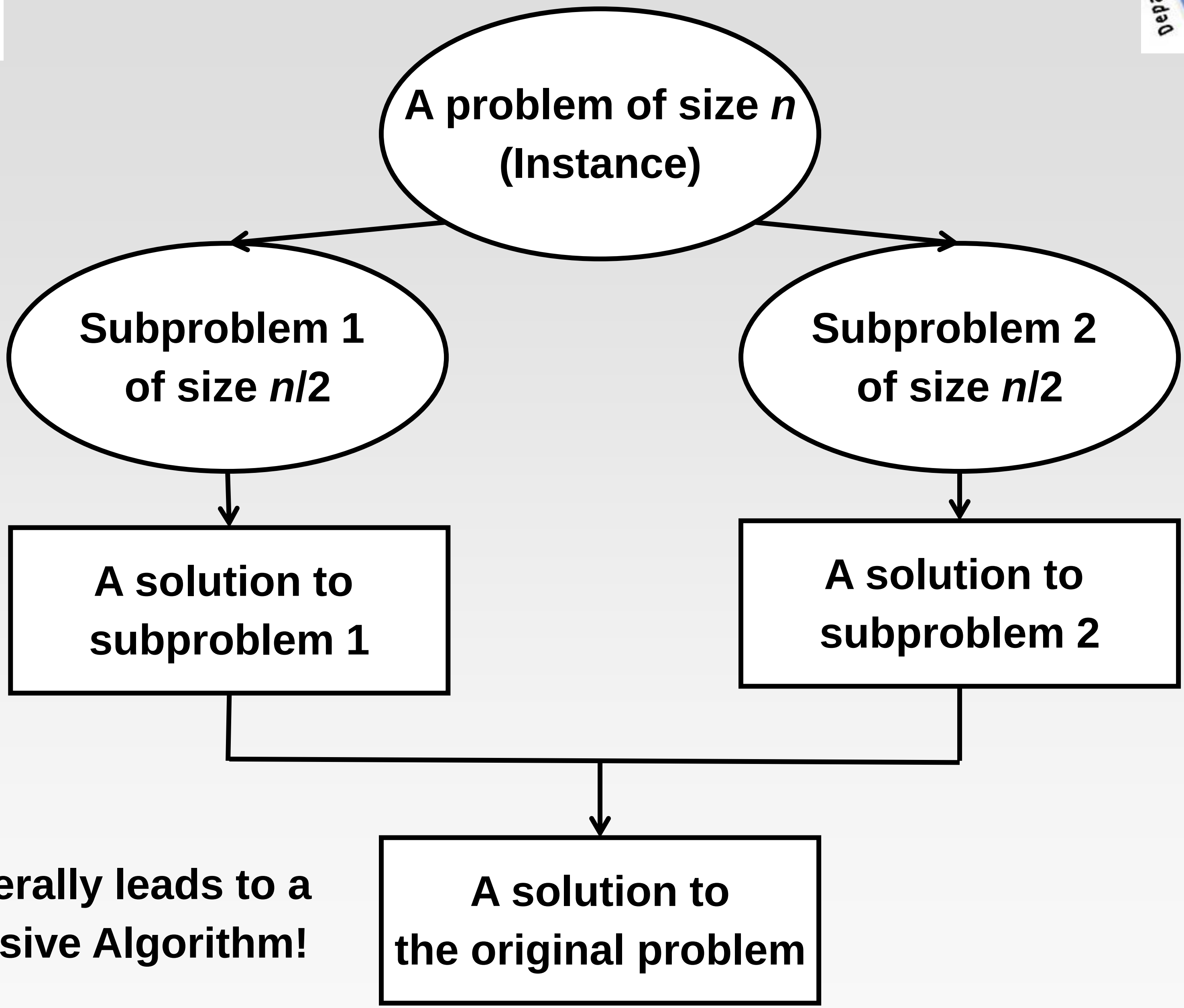
- **Binary Search**
- **Merge Sort**
- **Strassen's Matrix Multiplication.**



Divide-and-Conquer

- The most-well known algorithm design strategy.
- As its name implies **Divide-and-Conquer** involves dividing a problem into smaller problems that can be more easily solved.
- While the specifics vary from one application to another, divide-and-conquer always includes the following 3 steps in some form:
- **Divide** - Typically this steps involves splitting one problem into two problems of approximately ½ the size of original problem.
- **Conquer** - The divide step is repeated (usually recursively as subproblems are same type as original problem) until individual problem sizes are small enough to be solved (conquered) directly.
- **Recombine** - The solution to the original problem is obtained by combining all the solutions to the sub-problems.
- Divide and Conquer is not applicable to every problem class.
- Even when Divide and Conquer works it may not provide for an efficient solution.

Divide-and-Conquer Technique



It generally leads to a Recursive Algorithm!

Divide-and-Conquer

- **Control Abstraction** : A control abstraction is a procedure whose flow of control is clear but whose primary operations are specified by other procedures whose precise meanings are left undefined.
- The control abstraction for divide and conquer technique is DAndC (P), where P is the problem to be solved.

```
Algorithm DAndC (P)
{
  if Small(P) then return S(P);
else
{
  divide P into smaller instances P1, P2, …. Pk, k >= 1;
  apply DAndC to each of these sub problems;
  return (Combine (DAndC(P1), DAndC(P2),…., DAndC(Pk));
}
}
```

- For divide-and-conquer Small (P) is a Boolean valued function which determines whether the input size is small enough so that the answer can be computed without splitting.
- If this is so function ‘S’ is invoked otherwise, the problem ‘p’ into smaller sub problems.
- These sub problems $\mathbf{P_1, P_2, \ldots. P_k}$ are solved by recursive application of DAndC.
- Combine is a function that determines the solution to P using the solutions to the k subproblems.
- If the size of P is n and sizes of the k subproblems are $\boldsymbol{n_1, n_2, \ldots. n_k}$ then the computing time of DAndC is:

$$\mathbf{T}(\boldsymbol{n}) = \begin{cases} \boldsymbol{g(n)} & \boldsymbol{n\ small} \\ \boldsymbol{T(n_1) + T(n_2) + \ \ldots\ldots + T(n_k) + f(n)} & \boldsymbol{Otherwise} \end{cases}$$

- Where, **T(*n*)** is the time for DAndC on ‘*n*’ inputs,
- **g (*n*)** is the time to complete the answer directly for small inputs
- The function **f(*n*)** is the time for Dividing P and Combing solutions to subproblems.

- The complexity of many divide-and-conquer algorithms is given by recurrences of the form
- **T(*n*) =** $\begin{cases} T(1) & n = 1 \\ a\,T(n/b) + f(n) & n > 1 \end{cases}$
- Where, a and b are known constants.
- One of the method for solving any such recurrence relation is substitution method.
- For example consider the case where a = 2 and b = 2.
- Let T(1) = 2 and $f(n)$ = n, we have

$$T\,(n) = 2\,T(n/2) + n$$
$$= 2\,[\,2\,T(n/4) + n/2] + n$$
$$= 4\,T(n/4) + 2n$$
$$= 4\,[\,2\,T(n/8) + n/4] + 2n \quad ....$$
$$= 8\,T(n/8) + 3n$$

- In general, we see that

$$T\,(n) = 2^i\,T(n/2^i) + i\,n \qquad for\ any\ log_2\,n \geq i \geq 1$$

Binary Search

- Let $a_i, 1 \leq i \leq n$, be a list of elements that are sorted in order.
- When we are given a element 'x', binary search is used to find the corresponding element from the list.
- In case 'x' is present, we have to determine a value 'j' such that $a_j = x$ (successful search).
- If 'x' is not in the list then 'j' is to set to Zero (unsuccessful search).
- Divide-and-conquer can be used to solve this problem.
- Let Small(P) be true if $n = 1$.
- In this case, S(P) will take the value $i \; if \; x = ai$; otherwise it will take the value 0.
- If P has more than one element, it can divided (or reduced) into a new subproblem.

```
Algorithm BinSearch (a, n, x )
// Given an array a[1: n] of elements in increasing order,
// n ≥ 0, determine whether 'x' is present, and if so,
// return 'j' such that x = a[j] else return 0.
{
 low := 1; high := n;
  while (low ≤ high) do
  {
   mid := ⌊(low + high)/2⌋ ;
    if (x < a[mid]) then high := mid − 1;
    else if (x > a[mid]) then low := mid + 1;
            else return mid;
  }
   return 0;
 }
```

Example : Consider 14 elements

Index	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Elements	- 15	- 6	0	7	9	23	54	82	101	112	125	131	142	151
Comparisons	3	4	2	4	3	4	1	4	3	4	2	4	3	4

Low	High	Mid
Case 1 : Search $x = 151$		
1	14	7
8	14	11
12	14	13
14	14	14
		Found
Case 3 : Search $x = 9$		
1	14	7
1	6	3
4	6	5
		Found

Low	High	Mid
Case 2 : Search $x = -14$		
1	14	7
1	6	3
1	2	1
2	2	2
2	1	Not Found
Case 4 : Search $x = -43$		
1	14	7
1	6	3
1	2	1
1	0	Not Found

Binary Search

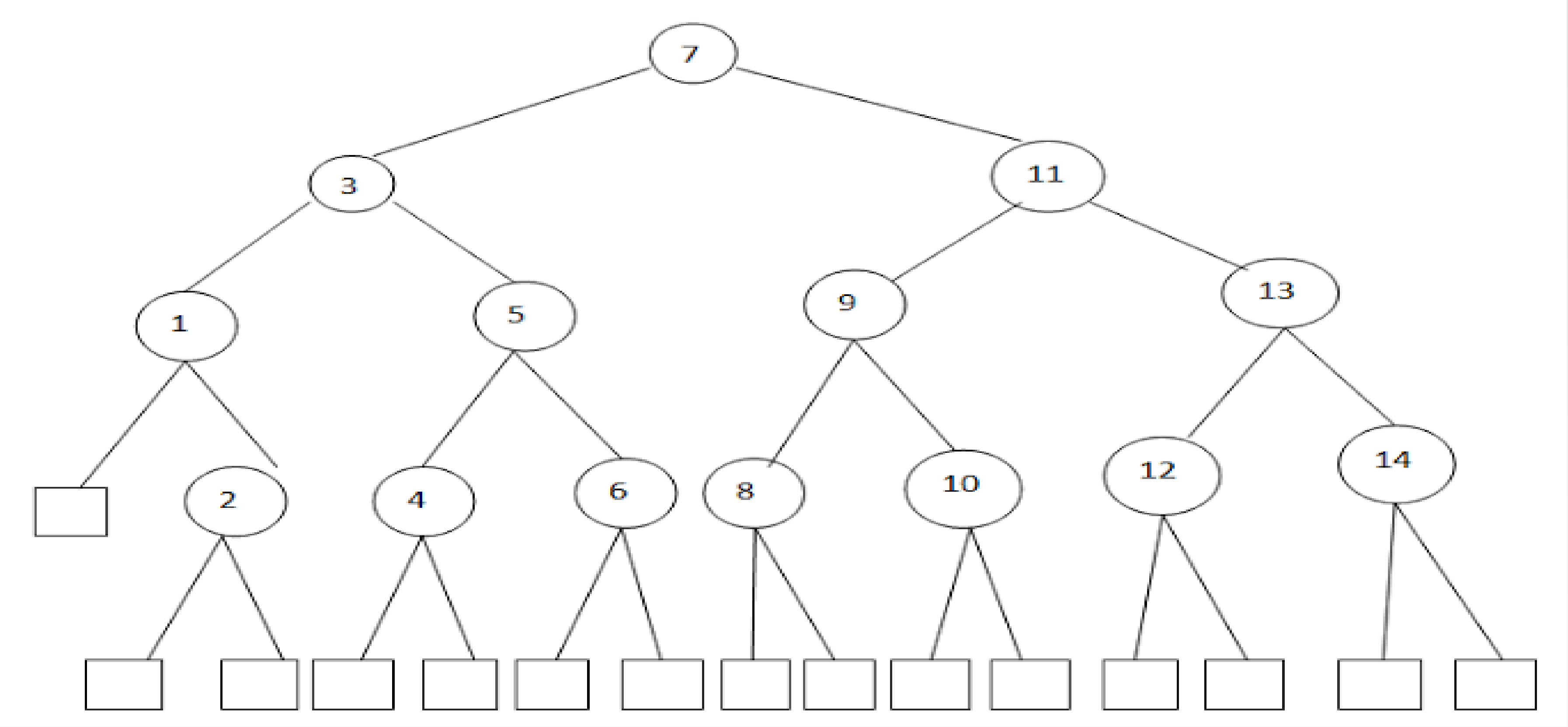
- It is found that No element requires more than 4 comparisons to be found (See Case 1 & 3).
- The average is obtained by summing the comparisons (Mentioned below elements) needed to find all 14 elements and dividing by 14.

$\mathbf{45/14 \approx 3.21}$ comparisons per successful search on average.

- There are 15 possible ways that an unsuccessful search may terminate depending on the value of $\boldsymbol{x}$.
- If $\boldsymbol{x < a[1]}$ the algorithm requires 3 elements comparison (See case 4) to determine that $\boldsymbol{x}$ is not present.
- For all remaining possibilities, BinSearch requires 4 elements comparison (See case 2) to determine that $\boldsymbol{x}$ is not present
- Thus the average number of elements comparisons for an unsuccessful search is $\mathbf{(3 + 14 \ * 4)\ /15 = 59\ /15 \approx 3.93}$.

Binary Search

- A better way to understand the algorithm is to consider the sequence of values for mid that are produced by BinSearch for all possible values of x.
- These values are nicely described using a binary decision tree.



Binary decision tree for binary search, $n = 14$

Binary Search

- If n is in the range of $[2^{k-1}, 2^k]$ then BinSearch makes at most k elements comparisons for a successful search and
- either $k - 1$ or k comparisons for an unsuccessful search.
- The computing time of binary search by giving formulas that describe the Best, Average, and Worst cases :

Successful Searches				**Unsuccessful Searches**
O (1)	**O ($\log n$)**	**O ($\log n$)**		**O ($\log n$)**
Best	**Average**	**Worst**		**Best, Average, Worst**

Merge Sort

- Merge Sort is an another classical example of a Divide-and-Conquer algorithm which has nice property that in the worst case its complexity is **O ($\boldsymbol{n}\log \boldsymbol{c}$)**.
- Given a sequence of $\boldsymbol{n}$ elements $\boldsymbol{a}[\mathbf{1}], \boldsymbol{a}[\mathbf{2}], \ldots\ldots \boldsymbol{a}[\boldsymbol{n}]$ to be sorted in increasing order.
- The merge sort algorithm divides input array in two sets, $\boldsymbol{a}[\mathbf{1}], \boldsymbol{a}[\mathbf{2}], \ldots\ldots \boldsymbol{a}[\lfloor \boldsymbol{n}/\mathbf{2} \rfloor]$ and $\boldsymbol{a}[\lfloor \boldsymbol{n}/\mathbf{2} \rfloor + \mathbf{1}], \ldots\ldots \boldsymbol{a}[\boldsymbol{n}]$
- And each set is individually sorted and resulting sorted sequences are merged to produce a single sorted sequence of $\boldsymbol{n}$ elements.
- Thus we can use Divide-and-Conquer strategy in which splitting is into two equal-sized sets and combining operation is merging of two sorted sets into one.

Merge Sort Algorithm

Algorithm MergeSort ($low, high$)
//$a[low : high]$ is a global array to be sorted.
{
  if ($low < high$) then // If there are more than one element
  {
    // Divide into subproblems.
    mid := $\lfloor (low + high)/2 \rfloor$; // Finds where to split the set
    MergeSort (low, mid**)** // Sort one subset
    MergeSort ($mid, high$**)** // Sort the other subset
    Merge ($low, mid, high$**)** // Combine the solutions
  }
}

Merge Algorithm

```
Algorithm Merge (low, mid, high)
// a [low : high] is a global array containing two sorted
// subsets in a [low : mid] and in a [mid + 1 : high].
// The objective is to merge these sorted sets into single
// sorted set a [low : high]. b[ ] is an auxiliary global array.
{
 h :=low; i := low; j:= mid + 1;
while ((h < mid) and (j < high)) do
 {
if (a[h] < a[j]) then
 { b[i] := a[h]; h := h + 1; }
 else
{b[i] :=a[j]; j := j + 1;}
 i := i + 1;
}
```

```
if (h > mid) then
 for k := j to high do
 {
b[i] := a[k]; i := i + 1;
}
else for k := h to mid do
{
 b[i] := a[k]; i := i + l;
}
for k := low to high do
   a[k] := b[k];
}
```

Merging two sorted subarrays using auxiliary storage

Merge Sort Example

- Consider an array of ten elements

  a[1:10] = (310, 285, 179, 652, 351, 423, 861, 254, 450, 520).
- Algorithm MergeSort begins by splitting a[ ] into two subarrays each of size five (a[1:5] and a[6:10]).
- The elements in a[1:5] are then split into two subarrays of size three (a[1:3]) and two (a[4:5]).
- Then the items in a[1:3] are split into subarrays of size two (a[1:2]) and one (a[3:3]).
- The two values in a[1:2] are split final time into one-element subarrays , and now the merging begins.
- A record of the subarrays is implicitly maintained by the recursive mechanism.

Merge Sort Example

- Pictorially the file can be viewed as

  (310 | 285 |179 | 652 , 351 | 423 , 861 , 254 , 450 , 520)

- where vertical bars indicate the boundaries of subarrays.
- Elements a[1] and a[2] are merged to yield

  (**285 , 310** |179 | 652 , 351 |423 , 861 , 254 , 450 , 520)

- Then a[3] is merged with a[1:2] and

  (**179 , 285 , 310** |652 , 351 | 423 , 861 , 254 , 450 , 520)

- Next elements a[4] and a[5] are merged:

  (179 , 285 , 310 |**351 , 652** | 423 , 861 , 254 , 450 , 520)

- And then a[1:3] and a[4:5]:

  (**179 , 285 , 310 , 351 , 652** | 423 , 861 , 254 , 450 , 520)

- At this point the algorithm has returned to the first invocation of MergeSoft and is about to process second recursive call.

Merge Sort Example

- Repeated recursive calls are invoked producing following subarrays

  (179 , 285 , 310 , 351 , 652 | 423 | 861 | 254 | 450 , 520)

- Elements a[6] and a[7] are merged. Then a[8] is merged with a[6:7]

  (179 , 285 , 310 , 351 , 652 | **254 , 423 , 861** | 450 , 520)

- Next a[9] and a[10] are merged and then a[6:8] and a[9:10]:

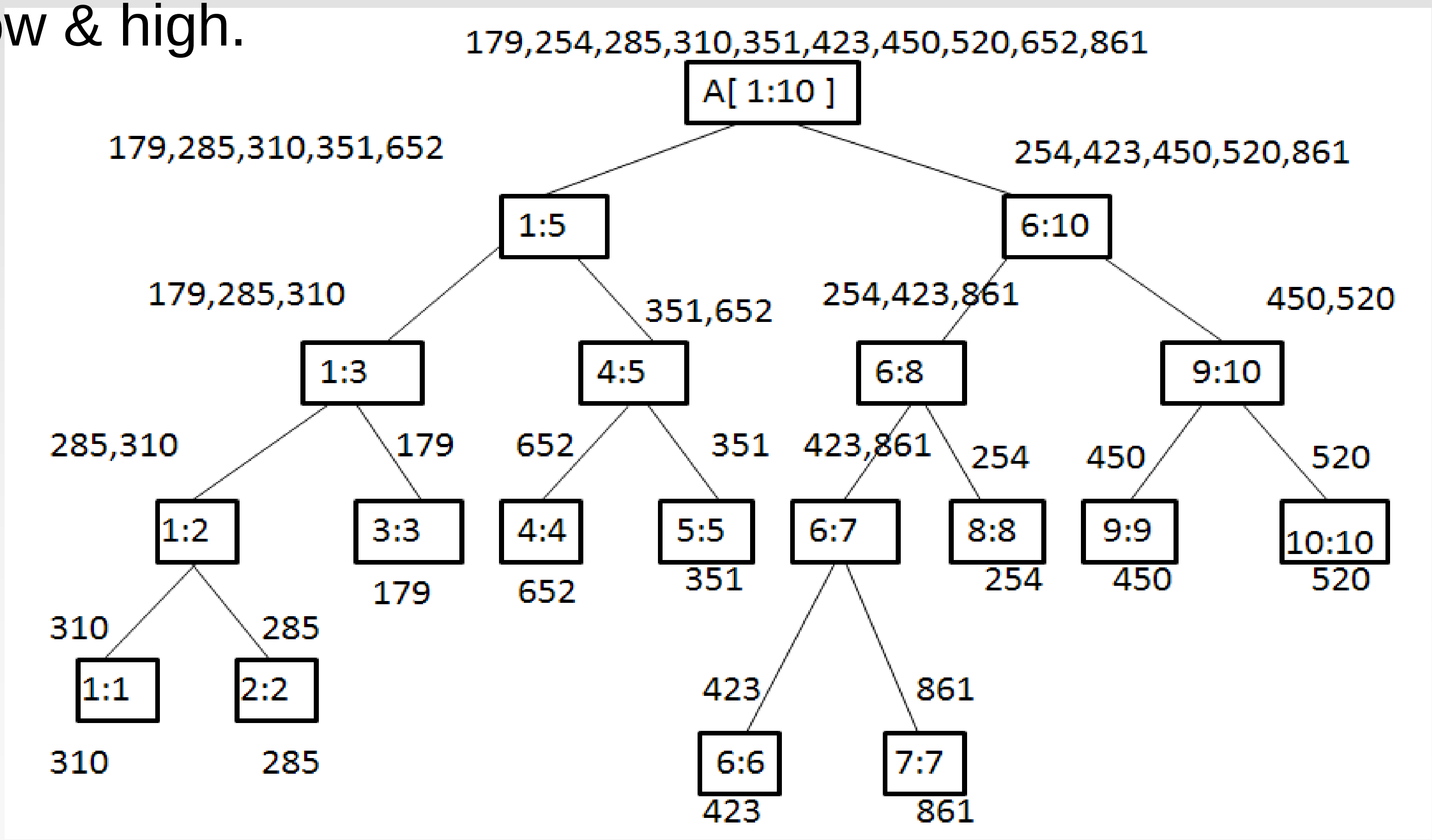
  (179 , 285 , 310 , 351 , 652 | **254, 423, 450, 520, 861**)

- At this point there are two sorted subarrays and the final merge produces fully sorted result.

  (179, 254, 285, 310, 351, 423, 450, 520, 652, 861)

Merge Sort Example

- The tree represents the sequence of recursive call that are produced by MergeSort.
- The pairs of values in each node are the values of parameters low & high.

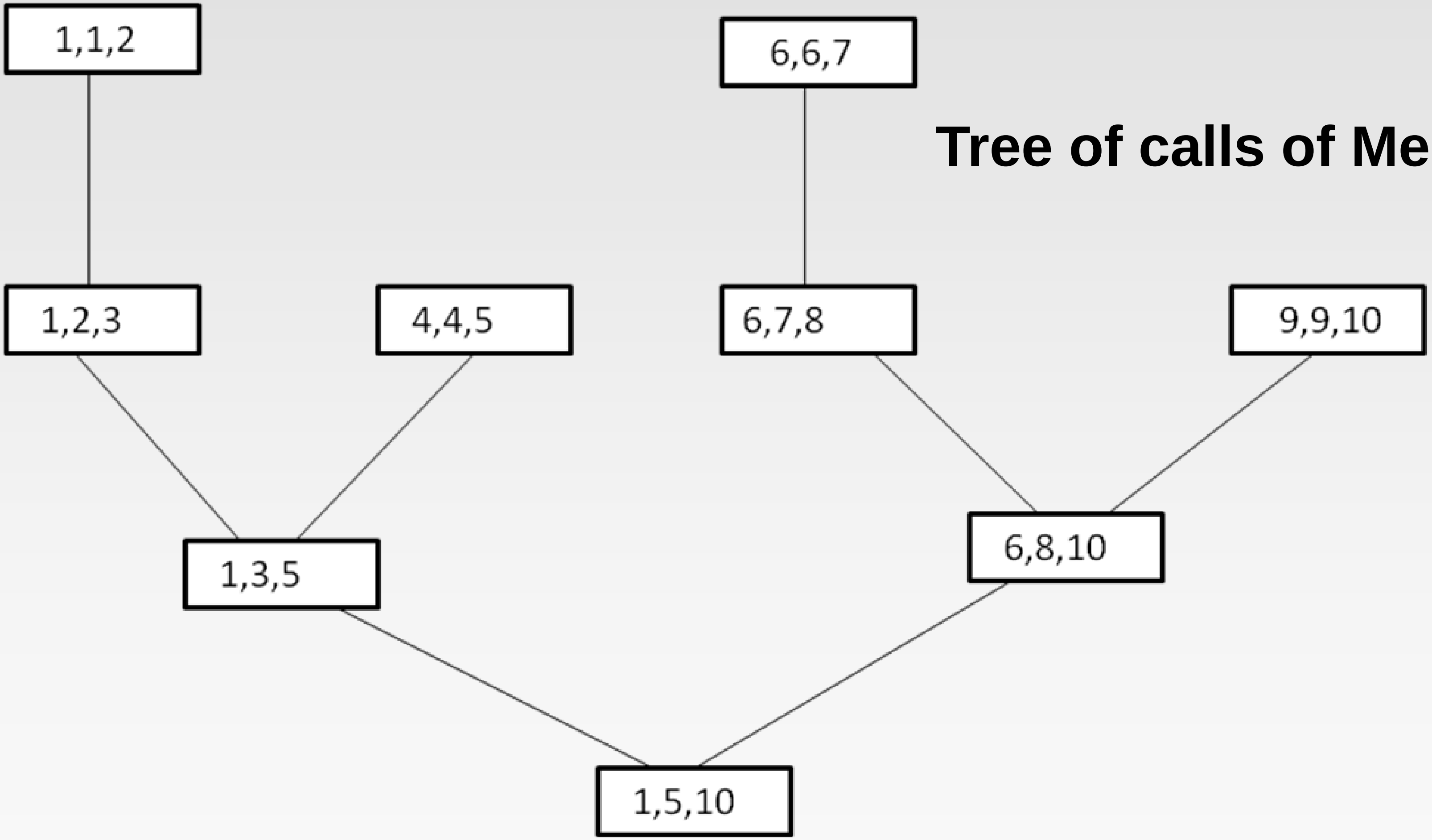


Tree of calls of MergeSort (1, 10)

Merge Sort Example

- The tree represents calls to procedure **Merge** by MergeSort.
- Reading Tree → Example : The node containing 1, 2, and 3 represents the merging of a[1:2] with a[3].

Tree of calls of Merge



Time Complexity of Merge Sort

- If the time for merging operation is proportional to n, then the computing time for merge sort is described by recurrence relation

$$T(n) = \begin{cases} a & n = 1, a \text{ is constant} \\ 2\,T(n/2) + c * n & n > 1, c \text{ is constant} \end{cases}$$

- When n is a power of 2, $n = 2^k$, (i.e. $k = \log n$) we can solve equation by substitutions

$$\begin{aligned} T(n) &= 2\,T(n/2) + c * n \\ &= 2\,T(2\,T(n/4) + c * n/2) + c * n \\ &= 4\,T(n/4) + 2\,c * n \\ &= 4\,(2\,T(n/8) + c * n/4) + 2 * c * n \\ &= 8\,T(n/8) + 3 * c * n \\ &\;\vdots \\ &= 2^k T(1) + k * c * n \\ &= a * n + c * n \log n \end{aligned}$$

- It is easy to see that is $2^k < n \leq 2^{k+1}$, $\textit{then } T(n) \leq T(2^{k+1})$.
- Therefore $T(n) = O(n \log n)$

Strassen's Matrix Multiplication

- Let A and B be two $\boldsymbol{n\ X\ n}$ matrices. The product matrix $\boldsymbol{C = AB}$ is calculated by using the formula,

  $$\boldsymbol{C(i,j) = \sum_{1 \le k \le n} A(i,k)B(k,j)}$$

- To compute $\boldsymbol{C(i,j)}$ we need $\boldsymbol{n}$ multiplications. As the matrix has $\boldsymbol{n^2}$ elements, the time complexity for the Matrix Multiplication is $\boldsymbol{O(n^3)}$
- The Divide-and-Conquer strategy suggest another way to find matrix multiplication of two $\boldsymbol{n\ X\ n}$ matrices.
- We assume that $\boldsymbol{n}$ is a power of 2, that is, there exists a integer $\boldsymbol{k}$ such that $\boldsymbol{n = 2^k}$
- Imagine that A and B are each partitioned into four square submatrices, each submatrix having dimension $\boldsymbol{\frac{n}{2}\ X\ \frac{n}{2}}$

Strassen's Matrix Multiplication

- If $n = 2$ then the product AB can be computed using above formula.

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Then

$$C_{11} = A_{11} B_{11} + A_{12} B_{21}$$
$$C_{12} = A_{11} B_{12} + A_{12} B_{22}$$
$$C_{21} = A_{12} B_{11} + A_{22} B_{21}$$
$$C_{22} = A_{21} B_{12} + A_{22} B_{22}$$

- For $n > 2$ the elements of C can be computed using matrix multiplication and addition operations applied to matrix of size $n/2 \ X \ n/2$.
- Since 'n' is a power of 2, these product can be recursively computed using the same formula .This Algorithm will continue applying itself to smaller sub matrix until 'n' become suitable small (n =2) so that the product is computed directly.

Strassen's Matrix Multiplication

- To compute $\boldsymbol{AB}$, we need eight multiplication and four additions of $\boldsymbol{n/2\ X\ n/2}$ matrices.
- Since two $\boldsymbol{n/2\ X\ n/2}$ matrices can be added in time $\boldsymbol{cn^2}$ for some constant c, the overall computing time $\boldsymbol{T(n)}$ of the resulting divide-and-conquer is given by the recurrence

$$\boldsymbol{T(n) = \begin{cases} b & n \leq 2 \\ 8\,T(n/2) + cn^2 & n > 2 \end{cases}}$$

- Where b and c are constants.
- This recurrence can be solved to obtain $\mathbf{T}(\boldsymbol{n}) = \boldsymbol{O}(\boldsymbol{n^3})$, hence no improvement over conventional method.
- Since matrix multiplications are more expensive than matrix additions, we can attempt to reformulate the equations for $\boldsymbol{C_{ij}}$ so as to have fewer multiplications and possibly more additions.

Strassen's Matrix Multiplication

- Volker Strassen has discovered a way to compute C_{ij} using only 7 multiplications and 18 additions.
- The idea of **Strassen's method** is to reduce the number of multiplications to 7.
- Strassen's method is similar to simple divide and conquer method in the sense that this method also divide matrices to sub-matrices of size $n/2 \ X \ n/2$, but in Strassen's method, the four sub-matrices of result are calculated using following formulae.

- His method involves first computing seven terms P, Q, R, S, T, U and V using 7 multiplications & 10 additions/subtractions.

$$P = (A_{11} + A_{22})(B_{11} + B_{22})$$
$$Q = (A_{21} + A_{22})B_{11}$$
$$R = A_{11}(B_{12} - B_{22})$$
$$S = A_{22}(B_{21} - B_{11})$$
$$T = (A_{11} + A_{12})B_{22}$$
$$U = (A_{21} - A_{11})(B_{11} + B_{12})$$
$$V = (A_{12} - A_{22})(B_{21} + B_{22})$$

- The C_{ij} require an additional 8 additions or subtractions.

$$C_{11} = P + S - T + V$$
$$C_{12} = R + T$$
$$C_{21} = Q + S$$
$$C_{22} = P + R - Q + U$$

- The resulting recurrence relation for $T(n)$ is

$$T(n) = \begin{cases} b & n \le 2 \\ 7\,T(n/2) + an^2 & n > 2 \end{cases}$$

where a and b are constants.

$$\begin{aligned}
T(n) &= 7\,T(n/2) + an^2 \\
&= 7\,(7\,T(n/4) + a(n/2)^2) + an^2 \\
&= 7^2 T(n/4) + a\,n^2(1 + \tfrac{7}{4}) \\
&= 7^3 T(n/8) + a\,n^2(1 + \tfrac{7}{4} + (\tfrac{7}{4})^2) \\
&= 7^k T(n/2^k) + a\,n^2\left(1 + \tfrac{7}{4} + \left(\tfrac{7}{4}\right)^2 \ldots. + \left(\tfrac{7}{4}\right)^{k-1}\right) \\
&(As\ n = 2^k \ \rightarrow \mathbf{k = log\ n\ \&\ T(1) = b}) \\
&\le 7^{log\,n}\, b + a\,n^2 \left(\tfrac{7}{4}\right)^{log\,n} \\
&\le b\,n^{log\,7} + a\,n^2\, n^{log(7/4)} \\
&\le b\,n^{log\,7} + a\,n^{2 + log\,7 - log\,4} \\
&\le b\,n^{log\,7} + a\,n^{log\,7}
\end{aligned}$$

$$\mathrm{T}(n) = O\left(n^{log\,7}\right) = O(n^{2.81})$$

Next - Unit III

Greedy Method